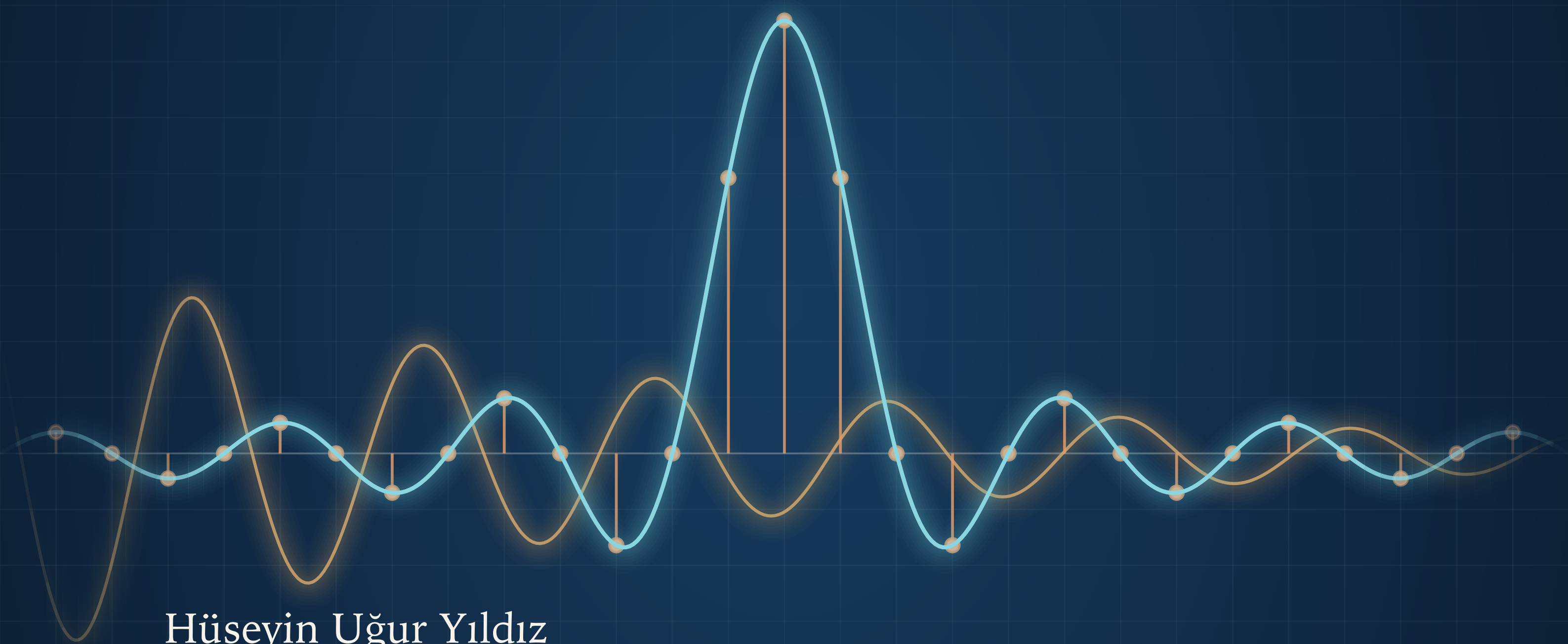
SIGNALS AND SYSTEMS

Signals, Systems and Frequency-Domain Analysis

Lecture Notes



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How to read these notes

These notes teach the full course at undergraduate level. They assume calculus, complex numbers and basic circuits. They define signals, classify systems and show why one function describes a linear time-invariant system. They then represent signals with complex exponentials and explain how sampling changes a signal.

Read the chapters in order because each chapter uses the one before it. First, a signal is defined as a function. Next, a system is defined as a map on functions. A linear time-invariant system is then described by one function. The frequency-domain chapters use this function to turn convolution into multiplication.

Each topic follows the same teaching order. A picture introduces the purpose of the idea. A definition and an equation state it exactly. A short derivation names each step. A worked example applies the method and ends with a check. Each worked example uses five headings: Given, Find, Method, Solution, Check. Try the Check step before you read it.

Two conventions apply throughout the notes. Energy and power are **normalised**: the resistance is $1\ \Omega$, so instantaneous power is $|x|^2$. The imaginary unit is j.

The third contents column gives textbook references. For example, **OW CH1.1–1.4** points to the matching material in Oppenheim and Willsky, *Signals and Systems*, second edition. The **OW** mark separates a textbook address from a course address. The numbering systems differ: these notes introduce the continuous-time Fourier transform in chapter 5, while the textbook introduces it in chapter 4.

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CHAPTER

1

Signals

A signal represents information as a function. This chapter fixes the notation for continuous-time and discrete-time signals. It then shows how to calculate energy and power, change the time axis, test periodicity and symmetry, and use impulses and complex exponentials. It ends with a catalogue of common signals.

1.1 Definitions and notation

A signal is a physical quantity that varies and carries information. Mathematically, it is a function of one or more independent variables.

The independent variable in this course is usually time. An image instead uses two space variables. The same signal operations apply to those variables.

Continuous-time and discrete-time signals use different domains. Keep their notation separate.

CONTINUOUS TIME

$$x(t), \qquad \forall t \in \mathbb{R}$$

Round brackets. The signal has a value at every real instant.

DISCRETE TIME

$$x[n], \qquad \forall n \in \mathbb{Z}$$

Square brackets. The signal has a value only at integer indices. Here n is an integer **index**, not a time in seconds.

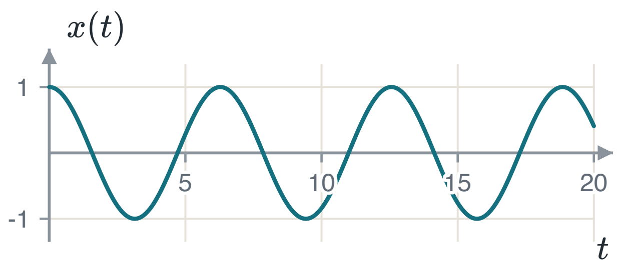

FIGURE 1.1 The continuous-time signal $x(t) = \cos(t)$ is drawn as an unbroken curve. It has a value at every real t.

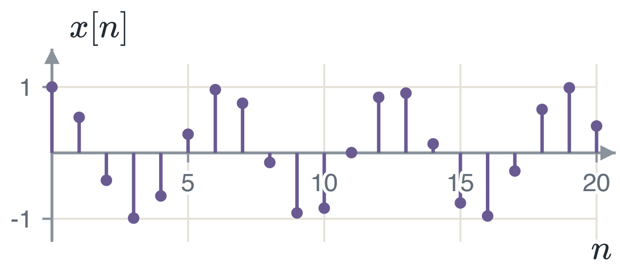

FIGURE 1.2 The discrete-time signal $x[n] = \cos(n)$ is drawn with stems. The dots are the signal values. A curve drawn through the dots is a different signal, a continuous-time one. Chapter 7 gives the conditions under which the samples fix that curve uniquely.

Two short cases show what the brackets mean. For $x(t) = \cos(\pi t)$, the value at $t = 0.5$ exists: $x(0.5) = \cos(\pi/2) = 0$. For $x[n] = n^2$, there is no value at $n = 1.5$, because 1.5 is not an integer. The sequence has the value $x[1] = 1$ at $n = 1$ and the value $x[2] = 4$ at $n = 2$, and nothing between them.

⊗ THE BRACKETS STATE THE DOMAIN

Writing $x[t]$ or $x(n)$ states the wrong domain. The domain determines the periodicity test, the convolution limits and the transform properties.

1.2 Energy and power

From circuit power to signal power

Signal energy and power come from circuit power. Let $v(t)$ be the voltage across a resistance R. The current is $i(t) = v(t)/R$ by Ohm's law, so the instantaneous power is

$$\begin{aligned} p(t) &= v(t)\, i(t) \\ &= v(t) \left(\frac{v(t)}{R} \right) \\ &= \frac{1}{R} v^2(t). \end{aligned}$$

Power is measured in watts. For example, a constant voltage $v(t) = 2$ V across $R = 4\ \Omega$ gives $p(t) = v^2/R = 4/4 = 1$ W at every instant.

Energy is power times time when the power is constant. The power of a signal varies, so integrate it over the time interval instead. The energy is the area under the power curve:

$$E = \int_{t_1}^{t_2} p(t)\,\mathrm{d}t = \int_{t_1}^{t_2} \frac{1}{R} v^2(t)\,\mathrm{d}t.$$

From this point onward, set $R = 1\ \Omega$ and use $|x(t)|^2$ as the power. This is the normalised convention. Restore the factor $1/R$ when a physical calculation uses a different resistance.

We use the modulus because a signal may be complex valued. The product $|x(t)|^2 = x(t)\,x^*(t)$ is real and never negative. Writing $x^2(t)$ instead is correct only for real signals.

Total energy

TOTAL ENERGY OVER ALL TIME

$$E_\infty \triangleq \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2\,\mathrm{d}t = \int_{-\infty}^{\infty} |x(t)|^2\,\mathrm{d}t$$

$$E_\infty \triangleq \lim_{N\to\infty} \sum_{n=-N}^{N} |x[n]|^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

The window is symmetric, $-T$ to T or $-N$ to N, so a two-sided signal is counted on both sides. Average power uses the same window.

For example, let $x[n] = 1$ for $n = 0, 1, 2$ and $x[n] = 0$ elsewhere. Only three terms of the sum are non-zero, so $E_\infty = |1|^2 + |1|^2 + |1|^2 = 3$.

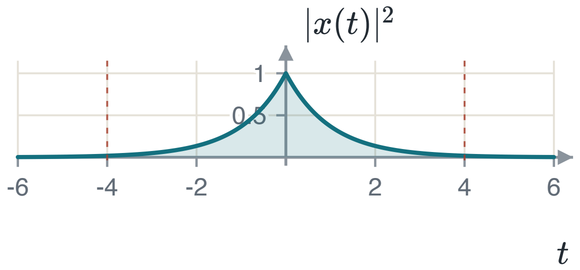

FIGURE 1.3 **Converging.** As the window $[-T, T]$ grows, the shaded area approaches a finite limit. The tails add less and less.

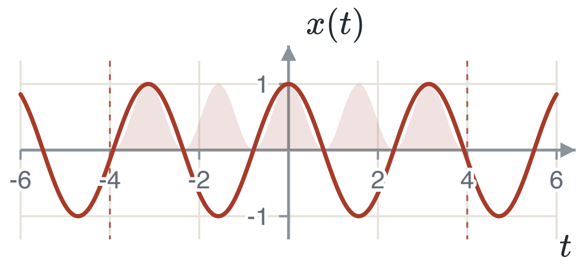

FIGURE 1.4 **Diverging.** For $x(t) = \cos(2t)$ every period adds the same area $\cos^2(2t)$, so $E_\infty \to \infty$.

When total energy diverges

The integral or the sum may not converge. Then the signal has no finite total energy. To see this, compute the energy E_T in the finite window $[-T, T]$ first, and take the limit last.

For $x(t) = \cos(2t)$, use the identity $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$ with $\theta = 2t$. Then integrate each term and evaluate at the limits:

$$\begin{aligned}
E_T &= \int_{-T}^{T} \cos^2(2t)\,\mathrm{d}t \\
&= \int_{-T}^{T} \frac{1+\cos(4t)}{2}\,\mathrm{d}t \\
&= \left[\frac{t}{2} + \frac{\sin(4t)}{8}\right]_{-T}^{T} \\
&= \left(\frac{T}{2} + \frac{\sin(4T)}{8}\right) - \left(-\frac{T}{2} - \frac{\sin(4T)}{8}\right) \\
&= T + \frac{\sin(4T)}{4}.
\end{aligned}$$

The fourth line uses $\sin(-4T) = -\sin(4T)$. The sine term stays between $-1/4$ and $1/4$, while T grows without limit. So $E_\infty \to \infty$. A constant does the same: for $x(t) = 2$, $E_T = \int_{-T}^{T} 2^2 \, \mathrm{d}t = 4 \cdot 2T = 8T$, which also grows without limit.

Such a signal still carries a finite amount of energy per unit of time. Average power measures that amount.

Average power

Average power is energy per unit time. Over a finite interval $[t_1, t_2]$, divide the energy by the length of the interval:

$$P = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} p(t) \, \mathrm{d}t.$$

For the average over all time, use the symmetric window $[-T, T]$ of length $2T$, as for the energy. Then let the window grow.

AVERAGE POWER OVER ALL TIME

$$P_\infty \triangleq \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, \mathrm{d}t$$

$$P_\infty \triangleq \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

In discrete time the divisor is $2N + 1$, because that is the number of samples from $-N$ to N with both ends included. A divisor of $2N$ gives the same limit, but the wrong value at a finite N.

For example, $x[n] = (-1)^n$ has $|x[n]|^2 = 1$ at every n. The window from $-N$ to N then holds energy $2N + 1$, so $P_\infty = \lim_{N \to \infty} (2N+1)/(2N+1) = 1$.

For $x(t) = \cos(2t)$, divide the window energy found above by the window length $2T$:

$$\begin{aligned} P_\infty &= \lim_{T \to \infty} \frac{E_T}{2T} \\ &= \lim_{T \to \infty} \frac{T + \sin(4T)/4}{2T} \\ &= \lim_{T \to \infty} \left(\frac{1}{2} + \frac{\sin(4T)}{8T} \right) \\ &= \frac{1}{2}. \end{aligned}$$

The last step uses $|\sin(4T)/(8T)| \le 1/(8T)$, which tends to 0.

⚠ TAKE THE LIMIT OF THE RATIO

Calculate the energy-to-duration ratio for each finite window first. Then take the limit. The ratio may approach any non-negative value, including zero.

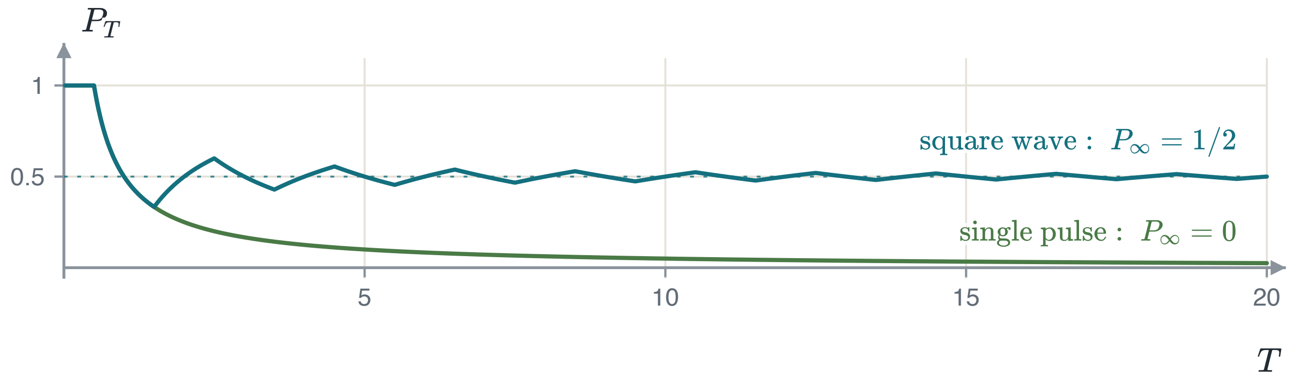

FIGURE 1.5 The running average $P_T = E_T/(2T)$ of two signals. The single pulse is 1 for $|t| < 1/2$ and has energy 1. The square wave repeats that pulse every 2 s. The two averages agree until $T = 3/2$. The square wave keeps adding energy, so its average approaches $1/2$. The pulse adds none, so its average $1/(2T)$ approaches 0.

The two limits in the figure follow from the definition. For the single pulse and every $T \geq 1/2$, the window holds the whole pulse, so $E_T = 1$ and $P_T = 1/(2T) \to 0$. For the square wave, take the windows with $T = 2k + \frac{1}{2}$, where k is a positive integer. Such a window holds the $2k+1$ whole pulses centred at $0, \pm 2, \ldots, \pm 2k$, each of energy 1:

$$P_T = \frac{E_T}{2T} = \frac{2k+1}{2\left(2k+\frac{1}{2}\right)} = \frac{2k+1}{4k+1} \longrightarrow \frac{1}{2} \quad \text{as } k \to \infty.$$

Energy signals, power signals, and neither

TABLE 1.1 Energy signals, power signals, and signals that are neither.

TYPE	CONDITION	TYPICAL SIGNALS
Energy signal	$E_\infty < \infty$, and then $P_\infty = 0$	Finite pulses and decaying responses
Power signal	$0 < P_\infty < \infty$, and then $E_\infty \to \infty$	Constants, sinusoids and other continuing signals
Neither	$E_\infty \to \infty$ and $P_\infty \to \infty$	Signals that grow without bound, such as $t\,u(t)$

The two energy-signal conditions are not independent. If E_∞ is finite, the energy in every finite window is at most E_∞. Dividing this bound by $2T$ makes P_∞ approach zero. The power condition follows from the energy condition:

$$\begin{aligned} E_T &= \int_{-T}^{T} |x(t)|^2 \, \mathrm{d}t \leq E_\infty \\ 0 \leq P_\infty &= \lim_{T\to\infty} \frac{E_T}{2T} \leq \lim_{T\to\infty} \frac{E_\infty}{2T} = 0 \\ \therefore \quad P_\infty &= 0. \end{aligned}$$

In the same way, a finite non-zero P_∞ forces $E_T \approx 2T\,P_\infty$ for large T, so the energy of a power signal diverges.

The constant $x(t) = 1$ is the simplest power signal. The energy in the window grows with the window, but the window length cancels in the average:

$$E_T = \int_{-T}^{T} 1^2 \, \mathrm{d}t = 2T \ \to \ \infty,$$
$$P_\infty = \lim_{T\to\infty} \frac{E_T}{2T} = \lim_{T\to\infty} \frac{2T}{2T} = 1.$$

A constant of another amplitude works the same way. For $x(t) = 3$, $|x(t)|^2 = 9$, so $E_T = 18T$ and $P_\infty = 18T/(2T) = 9$.

The ramp $x(t) = t$ for $t \geq 0$, and $x(t) = 0$ for $t < 0$, is in neither class. The lower limit of the energy integral is 0 because the ramp is zero for $t < 0$:

$$E_T = \int_{0}^{T} t^2 \, \mathrm{d}t = \left[\frac{t^3}{3}\right]_0^T = \frac{T^3}{3} \ \to \ \infty,$$
$$P_\infty = \lim_{T\to\infty} \frac{E_T}{2T} = \lim_{T\to\infty} \frac{T^3/3}{2T} = \lim_{T\to\infty} \frac{T^2}{6} = \infty.$$

The exponential $x(t) = e^t$ is also in neither class. Here $|x(t)|^2 = e^{2t}$, and

$$E_T = \int_{-T}^{T} e^{2t} \, \mathrm{d}t = \left[\frac{e^{2t}}{2}\right]_{-T}^{T} = \frac{e^{2T} - e^{-2T}}{2} \ \to \ \infty, \qquad P_\infty = \lim_{T\to\infty} \frac{e^{2T} - e^{-2T}}{4T} = \infty.$$

Both limits diverge. Compute both before you assign a class.

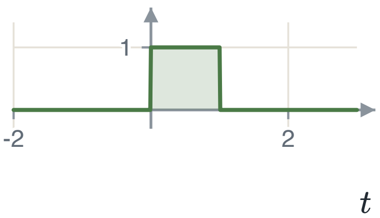

FIGURE 1.6 Energy signal: $E_\infty = 1$, $P_\infty = 0$.

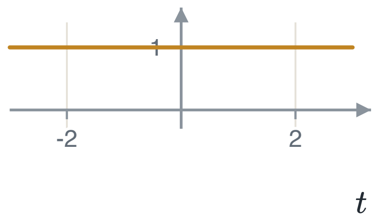

FIGURE 1.7 Power signal: $E_\infty \to \infty$, $P_\infty = 1$.

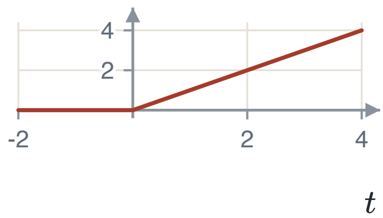

FIGURE 1.8 Neither: both limits diverge.

EXAMPLE 1.1

GIVEN $x(t) = 1$ for $0 \leq t \leq 1$, and $x(t) = 0$ otherwise.

FIND Is this an energy signal or a power signal?

METHOD Find E_∞ first because finite energy implies zero average power. This result then determines the class.

SOLUTION Start with the definitions and then use the support of the signal:

$$\begin{aligned} E_\infty &= \int_{-\infty}^{\infty} |x(t)|^2 \, \mathrm{d}t \\ &= \int_{0}^{1} 1^2 \, \mathrm{d}t \\ &= t\big|_0^1 = 1 < \infty. \end{aligned}$$

For every $T \geq 1$, the window contains the whole pulse. Therefore

$$\begin{aligned} P_\infty &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, \mathrm{d}t \\ &= \lim_{T\to\infty} \frac{1}{2T} \int_{0}^{1} 1 \, \mathrm{d}t \\ &= \lim_{T\to\infty} \frac{1}{2T} = 0. \end{aligned}$$

So $x(t)$ is an **energy signal**, with $E_\infty = 1$ J and $P_\infty = 0$ W. It is not a power signal.

CHECK Halving the amplitude must divide the energy by four. Direct calculation gives $\int_0^1 (1/2)^2 \, \mathrm{d}t = t/4|_0^1 = 1/4$, so the result has the required quadratic dependence on amplitude.

EXAMPLE 1.2

GIVEN $x[n] = 4$ for every integer n.

FIND Is this an energy signal or a power signal?

METHOD Each sample contributes the same positive energy, so E_∞ diverges. Infinite energy does not decide the class. Calculate P_∞ next and use the exact count $2N + 1$.

SOLUTION Each sample contributes $|4|^2 = 16$, so

$$\begin{aligned} E_\infty &= \sum_{n=-\infty}^{\infty} |x[n]|^2 \\ &= \sum_{n=-\infty}^{\infty} |4|^2 \\ &= \sum_{n=-\infty}^{\infty} 16 \to \infty. \end{aligned}$$

The finite window from $-N$ to N contains $2N + 1$ samples. Hence

$$\begin{aligned} P_\infty &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 \\ &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} 16 \\ &= \lim_{N\to\infty} \frac{(2N+1)16}{2N+1} \\ &= \lim_{N\to\infty} 16 = 16. \end{aligned}$$

So $x[n]$ is a **power signal**, with $P_\infty = 16$.

CHECK A constant of amplitude A must have $P_\infty = A^2$, and here $4^2 = 16$. The factor $2N + 1$ cancels exactly, which confirms the sample count was right.

> ⊗ DO NOT INFER POWER FROM ENERGY ALONE
>
> Infinite energy does not imply infinite average power. For the constant sequence, both the energy sum and the sample count grow at the same rate. Their ratio approaches 16.

A capacitor that discharges from 5 V is an everyday energy signal: $v(t) = 5e^{-t/\tau}$ for $t \geq 0$ and $v(t) = 0$ for $t < 0$, with time constant $\tau > 0$. The signal is zero for $t < 0$, so the energy integral starts at 0:

$$\begin{aligned} E_\infty &= \int_0^\infty \left(5e^{-t/\tau}\right)^2 \mathrm{d}t = \int_0^\infty 25\, e^{-2t/\tau}\, \mathrm{d}t \\ &= 25\left[-\frac{\tau}{2}e^{-2t/\tau}\right]_0^\infty = 25\left(0 + \frac{\tau}{2}\right) = \frac{25\tau}{2}. \end{aligned}$$

The mains voltage, $325\cos(2\pi\, 50\, t)$ V, never stops. Each period adds the same energy, so it is a power signal. A hand clap and the rainfall of one storm end, so they are energy signals.

1.3 Signal operations

This section changes the time axis of a signal in three ways: a shift, a reversal and a scaling. Each one changes the argument of x. None of them changes the height of the signal.

Time shifting

> TIME SHIFT
>
> $$x(t) \longrightarrow x(t - t_0)$$
>
> If $t_0 > 0$ the signal is **delayed** and moves right. If $t_0 < 0$ it is **advanced** and moves left.

Read the argument before moving the graph. At time t, the shifted signal uses the original value at $t - t_0$. For $t_0 > 0$, it reaches each original value t_0 seconds later. The graph therefore moves right. For example, $x(t+2)$ is $x(t - t_0)$ with $t_0 = -2$, so its graph is the graph of $x(t)$ moved 2 to the left.

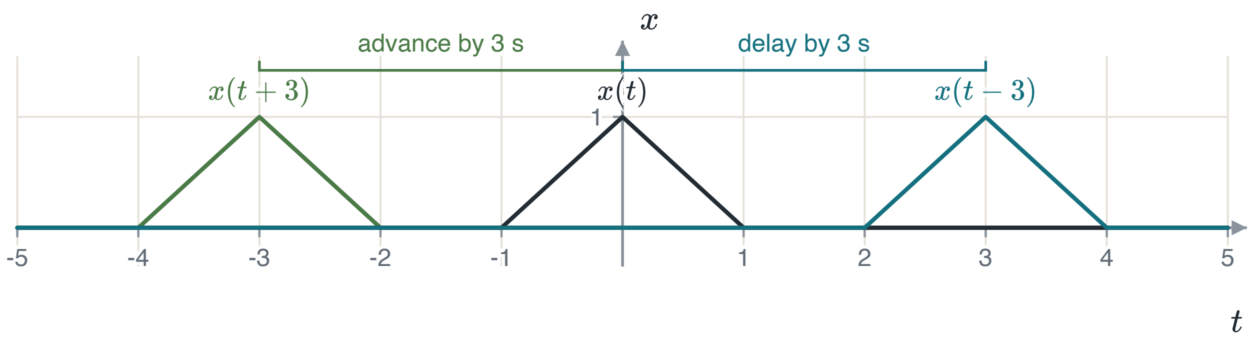

FIGURE 1.9 A time shift changes the signal position but not its shape.

For example, a radar sends a pulse $x(t)$ and the echo returns 0.2 ms later. With t in ms, the echo is $x(t - 0.2)$. It arrives later, so it is a delay with $t_0 = 0.2 > 0$. Thunder is the same: from a strike 1 km away it arrives about 3 s after the flash, so the sound we hear is $x(t-3)$.

TIME SHIFT IN DISCRETE TIME

$$x[n] \longrightarrow x[n-n_0], \qquad n_0 \in \mathbb{Z}$$

The sign rule is the same as in continuous time. If $n_0 > 0$ the stems move right. If $n_0 < 0$ they move left.

Every sample moves by the same whole number of steps. The shift must be an integer because a sequence has values only at integer indices. The expression $x[n-\frac{1}{2}]$ is therefore not defined.

Read a shifted sequence one sample at a time. Let $x[n] = 1 - n/4$ for $n = 0, 1, 2, 3$ and $x[n] = 0$ elsewhere, and let $y[n] = x[n-2]$. Then $y[3] = x[3-2] = x[1] = 1 - \frac{1}{4} = 0.75$.

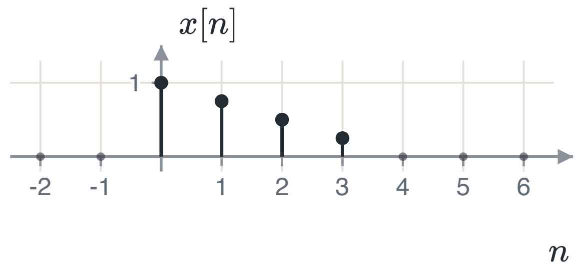

FIGURE 1.10 $x[n] = 1 - n/4$ for $n = 0, 1, 2, 3$.

FIGURE 1.11 $y[n] = x[n-2]$: every stem moves two samples right. The sample $x[1] = 0.75$ appears at $n = 3$.

A one-sample delay, $x[n-1]$, is the basic memory element of every difference equation in Chapter 3.

Time reversal

TIME REVERSAL

$$x(t) \longrightarrow x(-t) \qquad (x[n] \to x[-n])$$

Reflect the signal about the vertical axis. The value at time t moves to time $-t$.

Find the new support from the argument. Let x be non-zero only on $[\alpha, \beta]$. Then $x(-t)$ is non-zero only where $-t$ lies in $[\alpha, \beta]$. Multiply each inequality by -1. A negative multiplier reverses both inequalities:

$$\alpha \le -t \le \beta \iff -\beta \le t \le -\alpha.$$

So $x(-t)$ is non-zero on $[-\beta, -\alpha]$, and the width $\beta - \alpha$ does not change. A pulse on $[1, 3]$ moves to $[-3, -1]$ and keeps width 2. A signal that is non-zero only on $[-1, 4]$ becomes non-zero only on $[-4, 1]$.

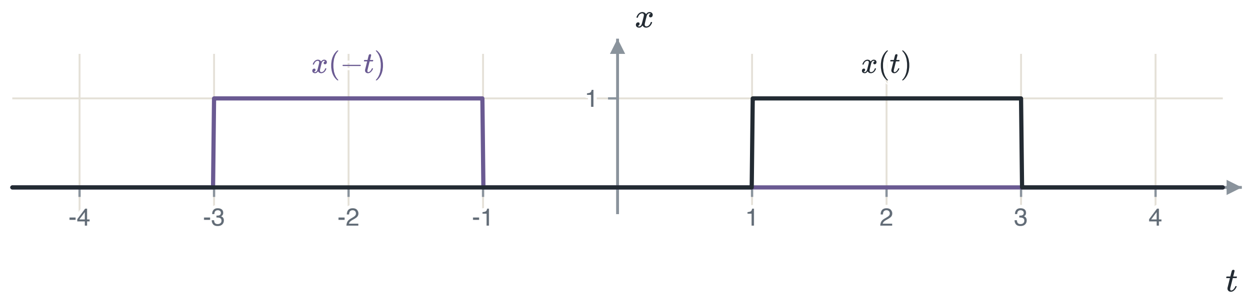

FIGURE 1.12 Reversal moves the pulse on $[1, 3]$ to $[-3, -1]$. The shape is kept; only the order in time changes.

This pulse has the same shape forwards and backwards, so reversal only moves it. A signal that is not symmetric in time changes more. A plucked guitar string at 220 Hz starts loud and dies away:

$$x(t) = e^{-4t}\Big[\sin(2\pi\,220\,t) + \tfrac{1}{2}\sin(2\pi\,440\,t) + \tfrac{1}{4}\sin(2\pi\,660\,t)\Big], \qquad t \geq 0,$$

and $x(t) = 0$ for $t < 0$. Played backwards, $x(-t)$ is also loudest at $t = 0$. It grows towards that instant and then stops at once. A steady tone does not change under reversal, because the cosine is even: $\cos(-2\pi\,440\,t) = \cos(2\pi\,440\,t)$.

Time scaling

TIME SCALING

$$y(t) = x(at), \qquad a > 0$$

If $a > 1$ the signal is compressed and speeded up. If $0 < a < 1$ it is stretched and slowed down. The height does not change.

Write the scaled signal as a new signal $y(t)$. Writing $x(t) = x(at)$ would force $a = 1$.

Find the new support by solving $at \in [\alpha, \beta]$. Dividing by $a > 0$ keeps the direction of both inequalities:

$$\begin{aligned} at &\in [\alpha, \beta] \\ \alpha \leq\ &at \leq \beta \\ \frac{\alpha}{a} \leq\ &t \leq \frac{\beta}{a}. \end{aligned}$$

So $x(at)$ is non-zero only on $[\alpha/a, \beta/a]$, and the width is divided by a. For example, if $x(t)$ is non-zero only on $[2, 6]$, then $y(t) = x(2t)$ needs $2 \leq 2t \leq 6$, that is $1 \leq t \leq 3$. The width falls from 4 to 2.

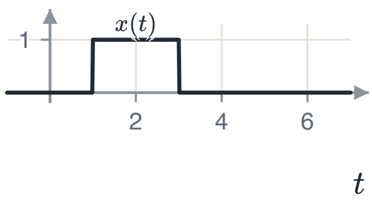

FIGURE 1.13 Original: support $[1, 3]$.

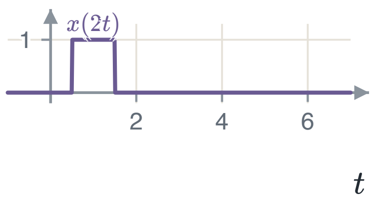

FIGURE 1.14 Compressed: support $[0.5, 1.5]$.

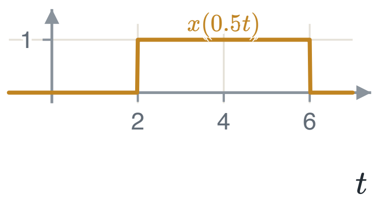

FIGURE 1.15 Stretched: support $[2, 6]$.

Time scaling also changes frequency. Substitute at for t in one tone:

$$x(t) = \sin(2\pi f_0 t) \quad\Longrightarrow\quad x(at) = \sin\big(2\pi f_0 (at)\big) = \sin\big(2\pi\,(af_0)\,t\big).$$

Every frequency is multiplied by a. Take a melody of four plucked notes, where note k starts at $t = 0.4k$ s for $k = 0, 1, 2, 3$, so the melody lasts 2 s. In $x(at)$, note k starts where $at = 0.4k$, that is at $t = 0.4k/a$ s. The melody then lasts $2/a$ s, and each pitch is multiplied by a. At $a = 2$ each frequency doubles, which is one octave up. This is what a voice message sounds like at double speed. At $a = \frac{1}{2}$, a 440 Hz tone becomes $\frac{1}{2} \cdot 440 = 220$ Hz.

> ⊗ DISCRETE-TIME SCALING NEEDS SEPARATE RULES
>
> $x[2n]$ keeps only the even-indexed samples $x[0], x[\pm 2], x[\pm 4], \ldots$ and discards the odd-indexed samples. This operation is decimation, and the lost samples cannot be recovered. A video played at double speed drops frames in the same way. The expression $x[n/2]$ is undefined at odd n unless an interpolation rule supplies those values. Continuous-time scaling does not have this integer-index restriction.

Combining a shift and a scale

To construct $x(at-b)$ from $x(t)$, use an intermediate signal so that each operation is explicit.

> SHIFT, THEN SCALE
>
> $$(1) \quad v(t) = x(t-b) \qquad\qquad (2) \quad y(t) = v(at) = x(at-b)$$
>
> Shift by b first. Then scale the result by a.

Step (2) replaces t by at in $v(t) = x(t-b)$, which gives $v(at) = x(at-b)$. For example, to get $x(2t-4)$, shift $x(t)$ right by 4 and then compress by 2. Shifting by 2 first would give $v(t) = x(t-2)$ and then $v(2t) = x(2t-2)$, which is a different signal.

> ⊗ WHY THE ORDER MATTERS
>
> Scaling first gives $w(t) = x(at)$. Shifting that by b gives $w(t-b) = x\big(a(t-b)\big) = x(at-ab)$. Unless $a = 1$ this is a different signal: it is shifted by b instead of by b/a.

EXAMPLE 1.3

GIVEN $x(t)$ is zero for $t < -2$, equal to 1 on $[-2, 0]$, equal to 2 on $[0, 2]$, and falls linearly from 2 to 0 on $[2, 4]$.

FIND Plot $y(t) = x(3t-5)$.

METHOD Here $a = 3$ and $b = 5$. Shift right by 5, then compress by 3. Each corner c of $x(t)$ moves to the time t that satisfies $3t - 5 = c$, and keeps its height.

SOLUTION For the shift, each original corner c moves to $c + 5$, so $-2, 0, 2, 4$ become $3, 5, 7, 9$. For the final signal, solve the argument equation for each original corner:

$$\begin{aligned} 3t - 5 &= c \\ 3t &= c + 5 \\ t &= \frac{c+5}{3}. \end{aligned}$$

Thus

$$c = -2, 0, 2, 4 \quad \Longrightarrow \quad t = \frac{3}{3}, \frac{5}{3}, \frac{7}{3}, \frac{9}{3} = 1, \frac{5}{3}, \frac{7}{3}, 3.$$

So $y(t)$ is 1 on $[1, \frac{5}{3}]$, 2 on $[\frac{5}{3}, \frac{7}{3}]$, falls linearly from 2 to 0 on $[\frac{7}{3}, 3]$, and is zero elsewhere.

CHECK The original support $[-2, 4]$ has width 6. Compression by $a = 3$ must give width $6/3 = 2$, and the result has support $[1, 3]$. Also set $3t - 5 = 0$. This gives $t = 5/3$, where $y(5/3) = x(0) = 2$.

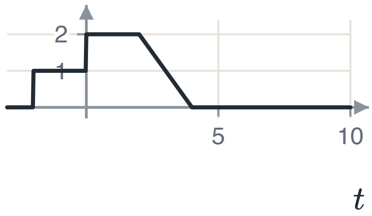

FIGURE 1.16 $x(t)$

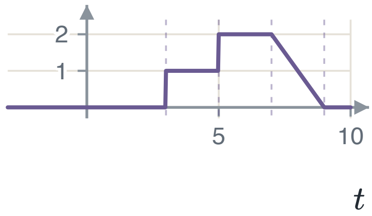

FIGURE 1.17 $v(t) = x(t-5)$

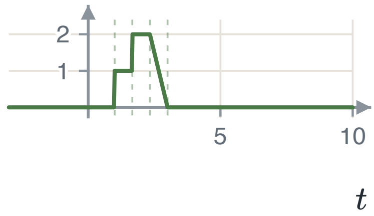

FIGURE 1.18 $y(t) = x(3t-5)$

1.4 Periodicity, even and odd

Periodic signals

PERIODICITY

$$x(t) = x(t+T) \quad \text{for all } t \in \mathbb{R}, \text{ for some } T > 0$$

$$x[n] = x[n+N] \quad \text{for all } n \in \mathbb{Z}, \text{ for some integer } N > 0$$

A signal that is not periodic is **aperiodic**.

The definition has two requirements. A positive period must exist, and the equality must hold for every t or n. Holding for some values is not enough. In continuous time, T may be any positive real number. In discrete time, N must be a positive integer because the sequence exists only at integer indices. A value such as 3.5 samples cannot be a period.

If T is a period then so is $2T$, $3T$, and so on. The **fundamental period** T_0 is the smallest positive period. The same applies to N_0 in discrete time.

FUNDAMENTAL FREQUENCY

$$\omega_0 = \frac{2\pi}{T_0} \qquad \text{and} \qquad \omega_0 = \frac{2\pi}{N_0}$$

Without the word *smallest*, ω_0 would not be well defined.

Two short cases show the definition at work. For $x(t) = \cos(\pi t/3)$, the angular frequency is $\omega_0 = \pi/3$, so $T_0 = 2\pi/(\pi/3) = 6$. For $x[n] = (-1)^n$, the test $N = 1$ fails because $(-1)^{n+1} = -(-1)^n$. The test $N = 2$ succeeds because $(-1)^{n+2} = (-1)^n(-1)^2 = (-1)^n$ for every n. So $N_0 = 2$.

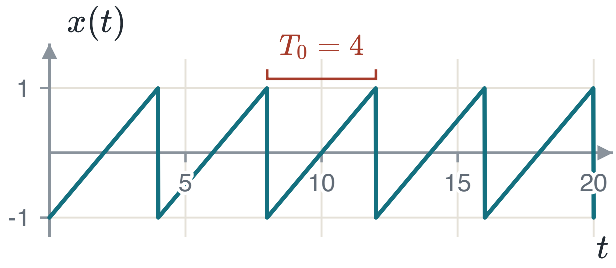

FIGURE 1.19 Periods are $4, 8, 12, \ldots$ The fundamental period is $T_0 = 4$.

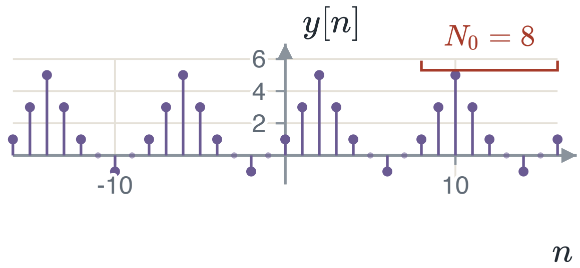

FIGURE 1.20 Periods are $8, 16, 24, \ldots$ The fundamental period is $N_0 = 8$.

A repeating shape that is not periodic

Each piece of a signal may repeat while the whole signal fails the test. Take

$$x(t) = \begin{cases} \sin(\pi t), & t < 0 \\ \cos(\pi t), & t \geq 0. \end{cases}$$

Both pieces have period 2. For $t \geq 0$, the test $x(t+T) = x(t)$ reads $\cos(\pi t + \pi T) = \cos(\pi t)$, so a period must be a multiple of 2: $T = 2m$ with $m = 1, 2, \ldots$ Test such a T at $t = -0.5$. The shifted point $-0.5 + 2m$ is positive, so the cosine piece applies there:

$$\begin{aligned} x(-0.5 + 2m) &= \cos\bigl(\pi(2m - 0.5)\bigr) = \cos(2\pi m - 0.5\pi) = \cos(0.5\pi) = 0, \\ x(-0.5) &= \sin(-0.5\pi) = -1. \end{aligned}$$

The two values differ for every m, so no T works and $x(t)$ is aperiodic. A shifted copy would carry the jump at $t = 0$ to $t = -T$, where $x(t)$ has no jump. The test must hold across the joint as well.

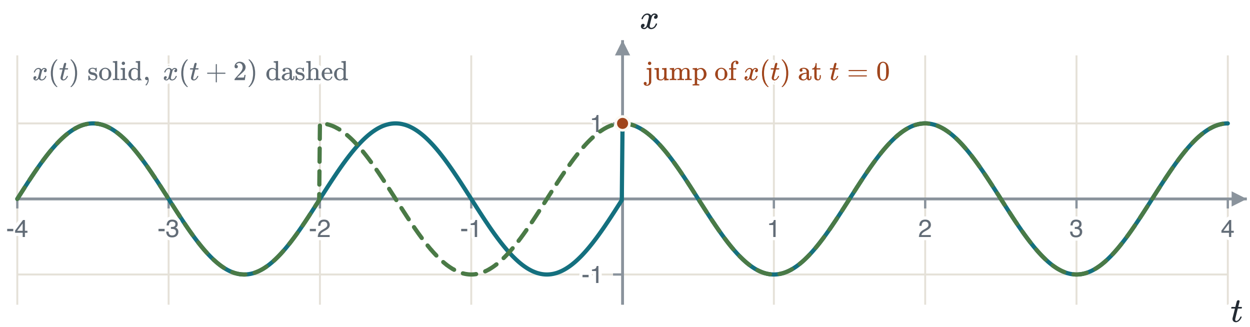

FIGURE 1.21 Both sides line up after a shift of 2, but the dashed copy $x(t+2)$ has its jump at $t = -2$, where $x(t)$ has none.

The signal $y(t) = \cos(\pi t)\, u(t)$ is aperiodic for the same reason. Suppose it had a period T. Then $y(t) = y(t - kT)$ for every positive integer k. Choose k so large that $t - kT < 0$; there y is zero, so $y(t) = 0$ for every t. But $y(0) = \cos 0 = 1$, so no period exists.

The period of a sum

Let $x(t) = x_1(t) + x_2(t)$, where x_1 has period T_1 and x_2 has period T_2. The sum repeats when both parts repeat at the same time. So look for a T that is a whole number of periods of each part, $T = kT_1 = mT_2$ with integers $k, m \geq 1$. Then

$$\begin{aligned} x(t+T) &= x_1(t+kT_1) + x_2(t+mT_2) \\ &= x_1(t) + x_2(t) = x(t). \end{aligned}$$

A COMMON PERIOD

$$T = k\,T_1 = m\,T_2 \quad \Longrightarrow \quad \frac{T_1}{T_2} = \frac{m}{k}$$

A common period exists only if T_1/T_2 is rational. The smallest one is the least common multiple of T_1 and T_2. It is the fundamental period of the sum unless terms cancel, which does not happen for a sum of sinusoids at different frequencies.

EXAMPLE 1.4

GIVEN $x(t) = \cos\left(\frac{2\pi t}{3}\right) + \sin\left(\frac{\pi t}{2}\right)$.

FIND The fundamental period T_0.

METHOD Find the period of each part from $T = 2\pi/\omega$. Then take the least common multiple from the prime factorisations.

SOLUTION The two periods are

$$\begin{aligned} T_1 &= \frac{2\pi}{2\pi/3} = 3, \\ T_2 &= \frac{2\pi}{\pi/2} = 4. \end{aligned}$$

Their ratio $3/4$ is rational. The factorisations are $3 = 3$ and $4 = 2^2$. They share no prime factor, so the least common multiple is their product:

$$T_0 = \mathrm{lcm}(3, 4) = 3 \cdot 4 = 12.$$

CHECK $12 = 4T_1 = 3T_2$. In one period of the sum the cosine makes four cycles and the sine makes three. No smaller common multiple exists: the multiples of 4 below 12 are 4 and 8, and neither is a multiple of 3.

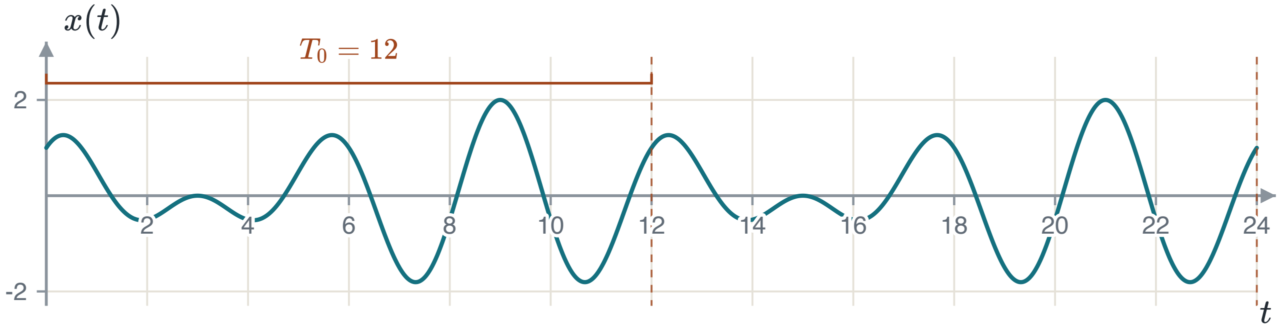

FIGURE 1.22 $x(t) = \cos\left(\frac{2\pi t}{3}\right) + \sin\left(\frac{\pi t}{2}\right)$ repeats every 12.

A second case: $x(t) = \cos(4t) + \cos(6t)$ has $T_1 = 2\pi/4 = \pi/2$ and $T_2 = 2\pi/6 = \pi/3$. The condition $k\,\pi/2 = m\,\pi/3$ gives $3k = 2m$, and the smallest positive solution is $k = 2$, $m = 3$. So

$T_0 = 2 \cdot \pi/2 = \pi$.

> ⚠ WHEN NO PERIOD EXISTS
>
> **Continuous time.** $\cos t + \cos(\sqrt{2}\,t)$ has $T_1 = 2\pi$ and $T_2 = 2\pi/\sqrt{2}$. The ratio $T_1/T_2 = \sqrt{2}$ is irrational, so no common period exists and the sum is aperiodic.
> **Discrete time.** A sum of two periodic sequences is always periodic. The periods N_1 and N_2 are integers, so $N = N_1 N_2$ is a whole number of periods of each part: $x_1[n + N_2 N_1] = x_1[n]$ and $x_2[n + N_1 N_2] = x_2[n]$.

Even and odd signals

A signal is **even** if $x(t) = x(-t)$ for every t, and **odd** if $x(t) = -x(-t)$ for every t. The same definitions apply to $x[n]$ with n in place of t. The graph of an even signal is its own mirror image about the axis $t = 0$. A half-turn about the origin maps the graph of an odd signal onto itself.

To test an odd signal at the origin, set $t = 0$ in the definition. This gives $x(0) = -x(0)$, so $2x(0) = 0$ and $x(0) = 0$. Therefore any signal with $x(0) \neq 0$ is not odd.

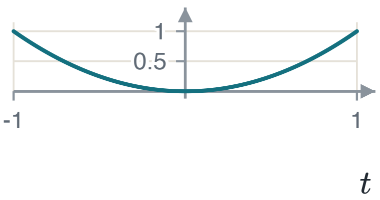

FIGURE 1.23 t^2 is even.

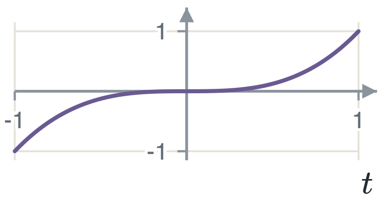

FIGURE 1.24 t^3 is odd.

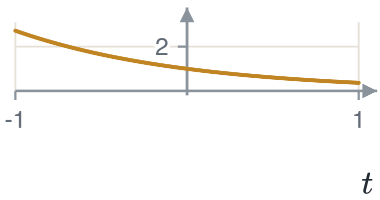

FIGURE 1.25 e^{-t} is neither.

To classify a signal, compute $x(-t)$ and compare it with $x(t)$. Do not rely on the look of the graph alone.

- $x(t) = \cos(\pi t)$ gives $x(-t) = \cos(-\pi t) = \cos(\pi t) = x(t)$, so it is even.
- $x(t) = \sin(\pi t)$ gives $x(-t) = \sin(-\pi t) = -\sin(\pi t) = -x(t)$, so it is odd.
- $x(t) = t \sin t$ gives $x(-t) = (-t)\sin(-t) = (-t)(-\sin t) = t \sin t = x(t)$, so it is even.
- $x(t) = e^{-2t}u(t)$ has $x(-1) = 0$ but $x(1) = e^{-2}$, so it is not even. It has $x(0) = 1 \neq 0$, so it is not odd. It is neither.
- $x[n] = n$ for $|n| \leq 3$ and 0 otherwise gives $x[-n] = -x[n]$ at every n, and $x[0] = 0$. So it is odd.

Every signal splits into an even part and an odd part:

> EVEN AND ODD PARTS
>
> $$\mathcal{E}\mathrm{v}\{x(t)\} = \tfrac{1}{2}\big[x(t) + x(-t)\big], \qquad \mathcal{O}\mathrm{dd}\{x(t)\} = \tfrac{1}{2}\big[x(t) - x(-t)\big]$$
>
> $$x(t) = \mathcal{E}\mathrm{v}\{x(t)\} + \mathcal{O}\mathrm{dd}\{x(t)\}$$
>
> The same construction works for $x[n]$.

Three short checks show that these definitions do what their names say. Replace t by $-t$ in each part. The even part does not change and the odd part changes sign:

$$\mathcal{E}\mathrm{v}\{x(-t)\} = \tfrac{1}{2}\big[x(-t) + x(t)\big] = \mathcal{E}\mathrm{v}\{x(t)\},$$
$$\mathcal{O}\mathrm{dd}\{x(-t)\} = \tfrac{1}{2}\big[x(-t) - x(t)\big] = -\mathcal{O}\mathrm{dd}\{x(t)\}.$$

Adding the two parts returns x, because the $x(-t)$ terms cancel:

$$\begin{aligned}\mathcal{E}\mathrm{v}\{x(t)\} + \mathcal{O}\mathrm{dd}\{x(t)\} &= \tfrac{1}{2}[x(t) + x(-t)] + \tfrac{1}{2}[x(t) - x(-t)] \\ &= \tfrac{1}{2}[2x(t)] \\ &= x(t).\end{aligned}$$

The split is unique. Suppose $x(t) = e(t) + o(t)$ with e even and o odd. Replacing t by $-t$ gives $x(-t) = e(t) - o(t)$. Adding and subtracting the two equations gives $e(t) = \frac{1}{2}[x(t) + x(-t)]$ and $o(t) = \frac{1}{2}[x(t) - x(-t)]$. These are the two formulas above.

Two quick cases. For $x(t) = t + 1$, $\mathcal{O}\mathrm{dd}\{x(t)\} = \frac{1}{2}(t+1) - \frac{1}{2}(-t+1) = t$ and $\mathcal{E}\mathrm{v}\{x(t)\} = 1$. For the step and $t \neq 0$, exactly one of $u(t)$ and $u(-t)$ is 1, so $\mathcal{E}\mathrm{v}\{u(t)\} = \frac{1}{2}[u(t) + u(-t)] = \frac{1}{2}$.

EXAMPLE 1.5

GIVEN The unit pulse $x(t) = 1$ for $0 < t < 1$, and $x(t) = 0$ otherwise.

FIND $\mathcal{E}\mathrm{v}\{x(t)\}$ and $\mathcal{O}\mathrm{dd}\{x(t)\}$.

METHOD Write down the mirror image $x(-t)$ first. Then apply the two definitions interval by interval.

SOLUTION The mirror image is $x(-t) = 1$ for $0 < -t < 1$, that is for $-1 < t < 0$, and 0 otherwise. The two pulses do not overlap, so on each interval only one of them is non-zero:

$$\mathcal{E}\mathrm{v}\{x(t)\} = \tfrac{1}{2}\big[x(t) + x(-t)\big] = \begin{cases} \frac{1}{2}(0+1) = \frac{1}{2}, & -1 < t < 0 \\ \frac{1}{2}(1+0) = \frac{1}{2}, & 0 < t < 1 \\ 0, & |t| > 1 \end{cases}$$

$$\mathcal{O}\mathrm{dd}\{x(t)\} = \tfrac{1}{2}\big[x(t) - x(-t)\big] = \begin{cases} \frac{1}{2}(0-1) = -\frac{1}{2}, & -1 < t < 0 \\ \frac{1}{2}(1-0) = \frac{1}{2}, & 0 < t < 1 \\ 0, & |t| > 1. \end{cases}$$

CHECK Add the two parts. On $0 < t < 1$: $\frac{1}{2} + \frac{1}{2} = 1 = x(t)$. On $-1 < t < 0$: $\frac{1}{2} - \frac{1}{2} = 0 = x(t)$. Outside $[-1, 1]$ both parts are 0. The sum returns $x(t)$ everywhere.

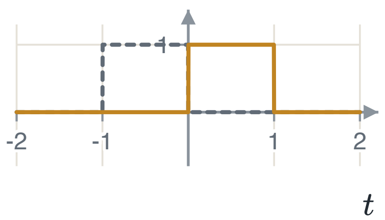

FIGURE 1.26 $x(t)$, with its mirror $x(-t)$ dashed.

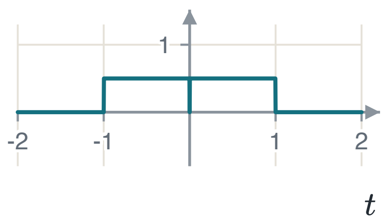

FIGURE 1.27 $\mathcal{E}\mathrm{v}\{x(t)\}$: height $\frac{1}{2}$ on both sides.

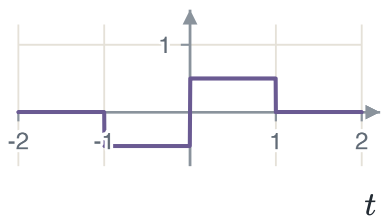

FIGURE 1.28 $\mathcal{O}\mathrm{dd}\{x(t)\}$: the left half flips below the axis.

Products of even and odd signals

Many integrals of a product are zero by symmetry alone. Two short results give this. Let x_e be even, let x_o be odd, and let $y(t) = x_e(t)\,x_o(t)$. Replace t by $-t$ and use the two definitions:

EVEN TIMES ODD IS ODD

$$y(-t) = x_e(-t)\,x_o(-t) = x_e(t)\big(-x_o(t)\big) = -y(t).$$

The same step shows that even times even is even, and odd times odd is even, because $(-1)(-1) = 1$.

Next, an odd signal integrates to zero over a symmetric interval $[-a, a]$. Split the integral at 0. In the left half put $t = -s$, so $\mathrm{d}t = -\mathrm{d}s$; the limit $t = -a$ becomes $s = a$ and the limit $t = 0$ becomes $s = 0$. Then use $x_o(-s) = -x_o(s)$:

$$\begin{aligned}\int_{-a}^{0} x_o(t)\,\mathrm{d}t &= \int_{a}^{0} x_o(-s)\,(-\mathrm{d}s)\\ &= \int_{a}^{0} x_o(s)\,\mathrm{d}s\\ &= -\int_{0}^{a} x_o(s)\,\mathrm{d}s,\\ \int_{-a}^{a} x_o(t)\,\mathrm{d}t &= \int_{-a}^{0} x_o(t)\,\mathrm{d}t + \int_{0}^{a} x_o(t)\,\mathrm{d}t\\ &= -\int_{0}^{a} x_o(s)\,\mathrm{d}s + \int_{0}^{a} x_o(t)\,\mathrm{d}t = 0.\end{aligned}$$

In discrete time, $\sum_{n=-N}^{N} x_o[n] = 0$ for the same reason: the terms at n and $-n$ cancel, and $x_o[0] = 0$. For example, t^3 is odd and $\cos t$ is even. Their product $t^3 \cos t$ is odd, so $\int_{-\pi}^{\pi} t^3 \cos t\,\mathrm{d}t = 0$ with no calculation. Chapter 4 uses this to find the Fourier coefficients of even and odd signals.

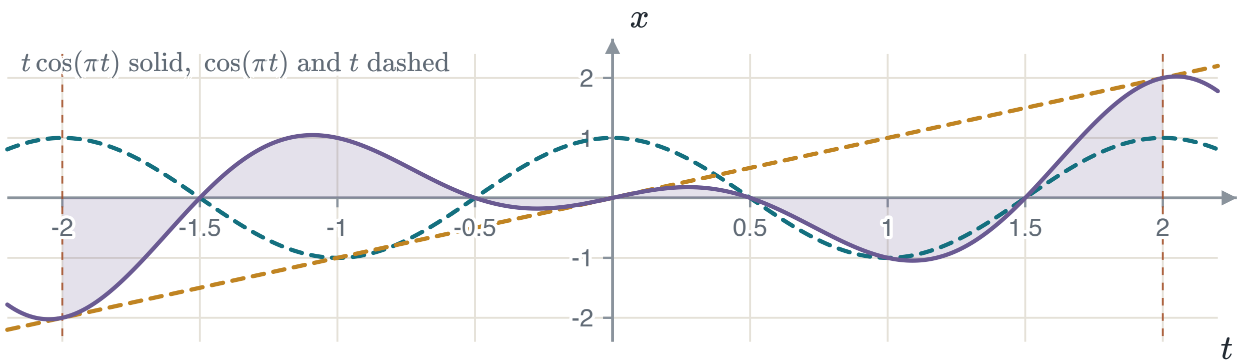

FIGURE 1.29 The even $\cos(\pi t)$ times the odd t gives the odd $t\cos(\pi t)$. Each shaded lobe on $[0, 2]$ has a lobe of opposite sign on $[-2, 0]$, so the area on $[-2, 2]$ is 0.

1.5 The impulse and the step

The discrete-time impulse and step

DEFINITIONS

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Both are ordinary sequences. Nothing infinite happens here.

A shifted impulse $\delta[n - n_0]$ is 1 only where $n - n_0 = 0$, that is at $n = n_0$. For example, $\delta[n-3]$ is non-zero only at $n = 3$. The step includes its first sample: $u[0] = 1$. So $x[n] = u[n-2]$ has $x[2] = u[0] = 1$.

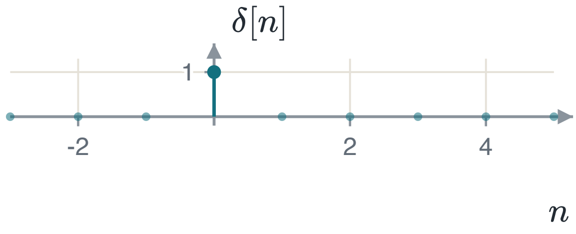

FIGURE 1.30 The unit sample $\delta[n]$: 1 at $n = 0$ and 0 elsewhere.

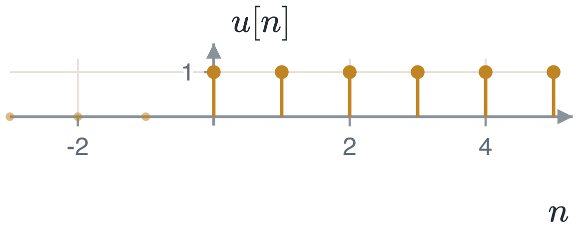

FIGURE 1.31 The unit step $u[n]$: 1 for $n \geq 0$, including $u[0] = 1$.

First difference and running sum

A first difference converts the step into the impulse. A running sum converts the impulse back into the step. These operations are the discrete-time forms of differentiation and integration.

FIRST DIFFERENCE AND RUNNING SUM

$$\delta[n] = u[n] - u[n-1]$$

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k] = \delta[n] + \delta[n-1] + \delta[n-2] + \cdots$$

Check the first difference region by region. For $n < 0$, both $u[n]$ and $u[n-1]$ are 0. At $n = 0$, $u[0] - u[-1] = 1 - 0 = 1$. For $n \geq 1$, both are 1, so they cancel. Only $n = 0$ survives, which is $\delta[n]$.

Each term $\delta[n-k]$ of the running sum places one unit sample at $n = k$. The terms $k = 0, 1, 2, \ldots$ therefore fill every $n \geq 0$ with a 1 and leave $n < 0$ at 0. To write the sum as an accumulation up to n, substitute $m = n - k$. Then $k = 0$ gives $m = n$, and $k \to \infty$ gives $m \to -\infty$:

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k] = \sum_{m=-\infty}^{n} \delta[m].$$

The two operations undo each other. A first difference reverses a running sum, and a running sum reverses a first difference. For example, take $x[n] = u[n] - u[n-3]$ region by region. For $n < 0$ it is $0 - 0 = 0$. For $n = 0, 1, 2$ it is $1 - 0 = 1$. For $n \geq 3$ it is $1 - 1 = 0$. So $u[n] - u[n-3] = \delta[n] + \delta[n-1] + \delta[n-2]$, with three non-zero samples.

The step as a sum of weighted impulses

The step can also be written with its own samples as weights:

$$u[n] = \sum_{k=-\infty}^{\infty} u[k]\, \delta[n-k].$$

Each term is an impulse at $n = k$ scaled by the sample $u[k]$. The weight $u[k]$ is 0 for $k < 0$ and 1 for $k \geq 0$, so it switches off every term with $k < 0$:

$$\begin{aligned} u[n] &= \sum_{k=-\infty}^{-1} 0 \cdot \delta[n-k] + \sum_{k=0}^{\infty} 1 \cdot \delta[n-k] \\ &= \sum_{k=0}^{\infty} \delta[n-k]. \end{aligned}$$

This is the running sum found above. A weighted sum of impulses is read one sample at a time. For $x[n] = 2\delta[n] + \delta[n-1]$, only the term $\delta[n-1]$ is non-zero at $n = 1$, so $x[1] = 1$.

Sampling and sifting

Sampling and sifting both use a shifted impulse, but they return different types of result.

SAMPLING PROPERTY

$$x[n]\, \delta[n-n_0] = x[n_0]\, \delta[n-n_0]$$

Both sides are **sequences**. Multiplying by a shifted impulse keeps one sample and deletes the rest.

Check it at each n. At $n = n_0$ both sides equal $x[n_0] \cdot 1$. At every other n, $\delta[n-n_0] = 0$, so both sides are 0.

SIFTING PROPERTY

$$x[n_0] = \sum_{n=-\infty}^{\infty} x[n]\, \delta[n-n_0]$$

The right-hand side is a **number**. Sifting is sampling followed by a sum.

To derive sifting, apply the sampling property inside the sum. Then take the constant $x[n_0]$ outside, and use the fact that $\delta[n-n_0]$ has exactly one non-zero sample:

$$\begin{aligned}\sum_{n=-\infty}^{\infty} x[n]\,\delta[n-n_0] &= \sum_{n=-\infty}^{\infty} x[n_0]\,\delta[n-n_0] \\ &= x[n_0] \sum_{n=-\infty}^{\infty} \delta[n-n_0] \\ &= x[n_0]\cdot 1.\end{aligned}$$

EXAMPLE 1.6

GIVEN $x[0]=1$, $x[1]=2$, $x[2]=3$, and $x[n]=0$ elsewhere. Take $n_0=2$.

FIND The sampled sequence and the sifted value.

METHOD Multiply by $\delta[n-2]$ to keep the sample at $n=2$. Then sum the sampled sequence to sift out its value.

SOLUTION Sampling:

$$x[n]\,\delta[n-2] = x[2]\,\delta[n-2] = 3\,\delta[n-2],$$

a single stem of height 3 at $n=2$. Sifting: only $n=0,1,2$ can give non-zero terms, and $\delta[n-2]$ is $0,0,1$ there:

$$\begin{aligned}\sum_{n=-\infty}^{\infty} x[n]\,\delta[n-2] &= x[0](0)+x[1](0)+x[2](1) \\ &= 1\cdot 0+2\cdot 0+3\cdot 1 \\ &= 3.\end{aligned}$$

CHECK The sampled result is a sequence. The sifted result is the number 3, which equals $x[2]$ as the sifting property requires.

The impulse may sit at a negative index. For $x[n]=n^2$, the impulse $\delta[n+1]=\delta[n-(-1)]$ sits at $n=-1$, so $\sum_n n^2\,\delta[n+1]=x[-1]=(-1)^2=1$.

Apply the sampling property at every integer shift k and add the results. This produces the representation used to derive convolution in Chapter 3.

REPRESENTATION PROPERTY

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

Any sequence is a sum of weighted, shifted impulses. The weights are the sample values themselves. At a fixed n, only the term $k=n$ is non-zero, and it equals $x[n]$.

The continuous-time impulse

The continuous-time step is $u(t)=1$ for $t>0$ and $u(t)=0$ for $t<0$. The continuous-time impulse is not an ordinary function.

DEFINE THE IMPULSE BY ITS ACTION

An ordinary function that is zero except at one point has integral 0, not 1. The impulse is therefore a **distribution**, also called a generalized function. A distribution is defined by how it acts inside an integral.

DEFINING PROPERTY (SIFTING)

$$x(t_0) = \int_{-\infty}^{\infty} x(t)\,\delta(t - t_0)\,\mathrm{d}t$$

This holds for every x that is continuous at t_0. It is the definition. Everything else about δ follows from it.

An informal picture is often written as $\delta(t) = \infty$ at $t = 0$, $\delta(t) = 0$ elsewhere, with $\int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}t = 1$. Use it only as a picture. A figure draws $\delta(t)$ as an arrow, and the number beside the arrow is its **weight**. The weight is an area, not a function value. So for $x(t) = 3\delta(t)$, $\int_{-\infty}^{\infty} x(t)\,\mathrm{d}t = 3$.

To picture the impulse, use a rectangle $\delta_\varepsilon(t)$ of width ε and height $1/\varepsilon$, centred at 0. Its area is $\varepsilon \cdot (1/\varepsilon) = 1$. A rectangle of width 0.1, for example, has height $1/0.1 = 10$. Now let $\varepsilon \to 0$. The rectangles do not approach an ordinary function at each fixed t. Their integrals against a continuous function do approach the sifting result. Shift the rectangle to t_0; it is $1/\varepsilon$ on $[t_0 - \varepsilon/2,\ t_0 + \varepsilon/2]$ and 0 elsewhere:

$$\int_{-\infty}^{\infty} x(t)\,\delta_\varepsilon(t - t_0)\,\mathrm{d}t = \frac{1}{\varepsilon}\int_{t_0-\varepsilon/2}^{t_0+\varepsilon/2} x(t)\,\mathrm{d}t \;\longrightarrow\; x(t_0) \quad \text{as } \varepsilon \to 0.$$

The middle expression is the average of x over a window of width ε around t_0. For x continuous at t_0, that average tends to the value at the centre.

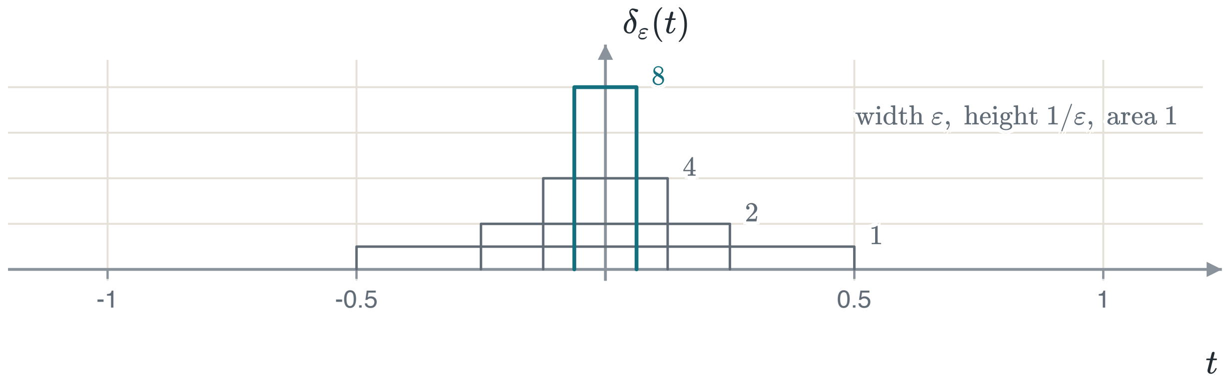

FIGURE 1.32 Unit-area rectangles with $\varepsilon = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$. The number beside each top is its height $1/\varepsilon$. As $\varepsilon \to 0$ the rectangle becomes the impulse $\delta(t)$.

Step, impulse, sampling and sifting in continuous time

The impulse and the step are related by differentiation and integration:

$$\delta(t) = \frac{\mathrm{d}}{\mathrm{d}t}u(t), \qquad u(t) = \int_{-\infty}^{t} \delta(\tau)\,\mathrm{d}\tau.$$

The running integral follows from the unit area of δ. For $t < 0$, the interval $(-\infty, t]$ does not contain the impulse at $\tau = 0$, so the integral is 0. For $t > 0$, the interval contains it, so the integral is its area, 1. That is $u(t)$. The pair $\delta[n] = u[n] - u[n-1]$ is the discrete-time analogue: the difference becomes a derivative and the sum becomes an integral.

SAMPLING PROPERTY, CONTINUOUS TIME

$$x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$$

Both sides are signals. Inside any integral they act in the same way, because the impulse sees only the value of x at t_0.

Sifting follows from sampling. Replace the product by $x(t_0)\,\delta(t-t_0)$, take the number $x(t_0)$ outside, and use the unit area:

$$\begin{aligned}\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,\mathrm{d}t &= \int_{-\infty}^{\infty} x(t_0)\,\delta(t-t_0)\,\mathrm{d}t \\ &= x(t_0)\int_{-\infty}^{\infty} \delta(t-t_0)\,\mathrm{d}t \\ &= x(t_0)\cdot 1.\end{aligned}$$

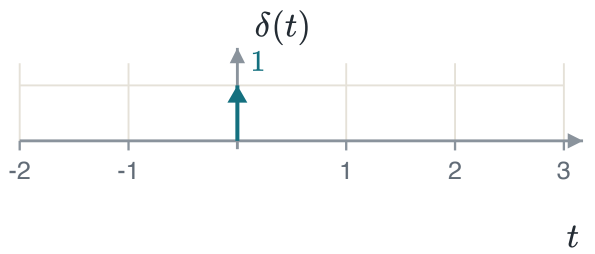

FIGURE 1.33 The arrow height shows the **area**, never a value of a function.

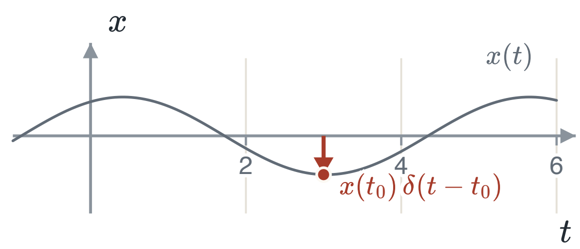

FIGURE 1.34 Sampling: the impulse at t_0 is scaled by the value of x there. Integrating returns that number.

EXAMPLE 1.7

GIVEN $x(t) = \cos t$ and $t_0 = \pi$.

FIND $$\int_{-\infty}^{\infty} \cos t\;\delta(t-\pi)\,\mathrm{d}t.$$

METHOD Use the sampling property to replace $\cos t$ by its value at the impulse. Then use the unit area of the impulse.

SOLUTION The impulse sits at $t = \pi$, so

$$\begin{aligned}\int_{-\infty}^{\infty} \cos t\;\delta(t-\pi)\,\mathrm{d}t &= \int_{-\infty}^{\infty} \cos(\pi)\,\delta(t-\pi)\,\mathrm{d}t \\ &= \cos(\pi)\int_{-\infty}^{\infty} \delta(t-\pi)\,\mathrm{d}t \\ &= (-1)(1) = -1.\end{aligned}$$

CHECK The result is a number, not a signal. It equals $x(t_0) = \cos\pi = -1$, as the sifting property requires.

Scaling the impulse

Compressing the time axis changes the weight of an impulse, not its position. Find the area of $\delta(at)$ with the substitution $s = at$, so $\mathrm{d}t = \mathrm{d}s/a$. For $a > 0$ the limits stay $-\infty$ and ∞:

$$\int_{-\infty}^{\infty} \delta(at)\,\mathrm{d}t = \int_{-\infty}^{\infty} \delta(s)\,\frac{\mathrm{d}s}{a} = \frac{1}{a}\int_{-\infty}^{\infty} \delta(s)\,\mathrm{d}s = \frac{1}{a}.$$

For $a < 0$ the substitution swaps the limits: $t = -\infty$ gives $s = +\infty$, and $t = +\infty$ gives $s = -\infty$. Swapping them back changes the sign:

$$\int_{-\infty}^{\infty} \delta(at)\,\mathrm{d}t = \int_{\infty}^{-\infty} \delta(s)\,\frac{\mathrm{d}s}{a} = -\frac{1}{a}\int_{-\infty}^{\infty} \delta(s)\,\mathrm{d}s = -\frac{1}{a} = \frac{1}{|a|}.$$

The same substitution inside a sifting integral gives $\int x(t)\,\delta(at)\,\mathrm{d}t = x(0)/|a|$, because $x(s/a)$ equals $x(0)$ at $s = 0$. So $\delta(at)$ acts exactly like $\delta(t)/|a|$:

SCALING PROPERTY

$$\delta(at) = \frac{1}{|a|}\,\delta(t), \qquad a \neq 0$$

The impulse stays at $t = 0$. Only its weight changes. In the rectangle picture, $\delta_\varepsilon(at)$ keeps its height and has width $1/|a|$ times the original, so its area is $1/|a|$.

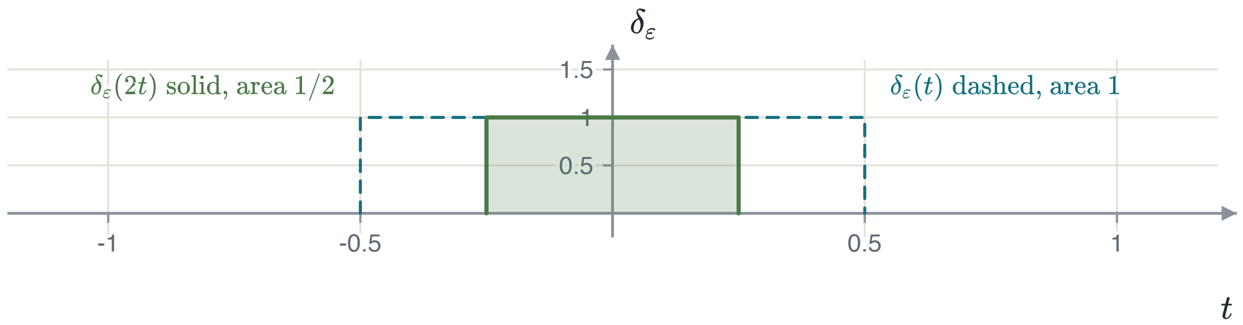

FIGURE 1.35 A unit-area rectangle of width 1 and height 1, and the same rectangle with t replaced by $2t$. The height stays 1 and the width halves, so the area halves.

EXAMPLE 1.8

GIVEN $x(t) = \cos t$.

FIND $$\int_{-\infty}^{\infty} \cos t\;\delta(2t)\,\mathrm{d}t.$$

METHOD Use the scaling property to rewrite $\delta(2t)$ as a weighted impulse at $t = 0$. Then sift.

SOLUTION With $a = 2$, $\delta(2t) = \frac{1}{2}\delta(t)$. Then

$$\int_{-\infty}^{\infty} \cos t\, \delta(2t)\, \mathrm{d}t = \frac{1}{2}\int_{-\infty}^{\infty} \cos t\, \delta(t)\, \mathrm{d}t = \frac{1}{2}\cos 0 = \frac{1}{2}.$$

CHECK Substitute $s = 2t$ directly, with $\mathrm{d}t = \mathrm{d}s/2$ and unchanged limits: $\int \cos(s/2)\,\delta(s)\,\mathrm{d}s/2 = \frac{1}{2}\cos 0 = \frac{1}{2}$. The two routes agree.

Differentiating a signal with jumps

A signal with jumps has a derivative in two parts. On each smooth piece, differentiate as usual. At a jump of size k at t_0, the signal contains a step $k\,u(t - t_0)$, and its derivative is $k\,\delta(t - t_0)$. So each jump adds an impulse whose weight is the jump.

EXAMPLE 1.9

GIVEN $x(t) = t$ on $[0, 2]$, $x(t) = 1$ on $(2, 3)$, and $x(t) = 0$ elsewhere.

FIND $x'(t)$.

METHOD (1) Differentiate each piece: the ramp gives 1 and a constant gives 0. (2) At each jump of size k at t_0, add $k\,\delta(t - t_0)$. (3) Check that the running integral of the result returns $x(t)$.

SOLUTION The pieces give slope 0 for $t < 0$, slope 1 for $0 < t < 2$, and slope 0 for $2 < t < 3$ and for $t > 3$. The slope 1 on $(0, 2)$ is $u(t) - u(t - 2)$. Now read the jumps as the value just after minus the value just before:

$$\begin{aligned} t = 0: &\quad x(0^+) - x(0^-) = 0 - 0 = 0, \\ t = 2: &\quad x(2^+) - x(2^-) = 1 - 2 = -1, \\ t = 3: &\quad x(3^+) - x(3^-) = 0 - 1 = -1. \end{aligned}$$

The ramp starts at 0, so there is no impulse at $t = 0$. Therefore

$$x'(t) = u(t) - u(t - 2) - \delta(t - 2) - \delta(t - 3).$$

CHECK Integrate x' from $-\infty$ to t. For $0 < t < 2$: $\int_0^t 1\,\mathrm{d}\tau = t = x(t)$. For $2 < t < 3$: $\int_0^2 1\,\mathrm{d}\tau - 1 = 2 - 1 = 1 = x(t)$. For $t > 3$: $2 - 1 - 1 = 0 = x(t)$.

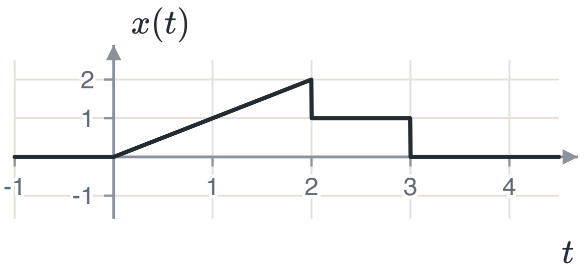

FIGURE 1.36 $x(t)$: a ramp to 2, a drop to 1 at $t = 2$, and a drop to 0 at $t = 3$.

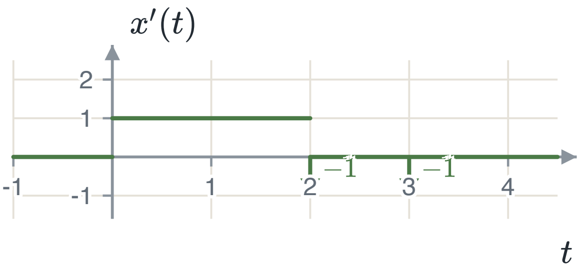

FIGURE 1.37 $x'(t)$: slope 1 on $(0, 2)$ and two impulses of weight -1, drawn pointing down.

1.6 Complex exponentials

Complex numbers in polar form

A complex exponential is easiest to read when its complex constants are in polar form. A complex number has two forms, linked by Euler's relation $e^{j\theta} = \cos\theta + j\sin\theta$:

> TWO FORMS OF ONE NUMBER
>
> $$\begin{aligned} z &= x + jy \\ &= re^{j\theta} \\ &= r\cos\theta + jr\sin\theta \end{aligned}$$
>
> $$r = |z| = \sqrt{x^2 + y^2}, \qquad \theta = \angle z$$
>
> Read θ from the quadrant of the point (x, y). The value $\arctan(y/x)$ alone cannot tell $1 + j$ from $-1 - j$: both have $y/x = 1$, but $1 + j = \sqrt{2}\, e^{j\pi/4}$ and $-1 - j = \sqrt{2}\, e^{-j3\pi/4}$.

Comparing the real parts and the imaginary parts of the two forms gives $x = r\cos\theta$ and $y = r\sin\theta$. For example, $z = 2e^{j\pi/3}$ has $x = 2\cos(\pi/3) = 1$ and $y = 2\sin(\pi/3) = \sqrt{3}$, so $z = 1 + j\sqrt{3} \approx 1 + j1.73$. In the other direction, $z = -2j$ has $x = 0$ and $y = -2$. Then $r = \sqrt{0^2 + (-2)^2} = 2$, and the point lies on the negative imaginary axis, so $\theta = -\pi/2$ and $-2j = 2e^{-j\pi/2}$.

Products are simple in polar form. The law of exponents gives $r_1 e^{j\theta_1}\, r_2 e^{j\theta_2} = r_1 r_2\, e^{j(\theta_1 + \theta_2)}$: the moduli multiply and the angles add.

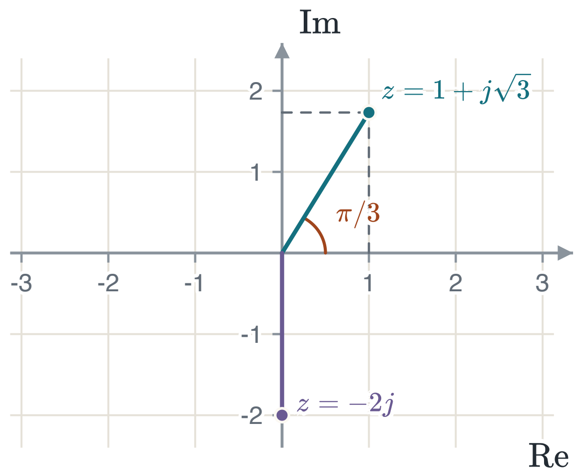

FIGURE 1.38 $z = 2e^{j\pi/3} = 1 + j\sqrt{3}$ lies at distance $r = 2$ and angle $\pi/3$. $z = -2j = 2e^{-j\pi/2}$ lies on the negative imaginary axis.

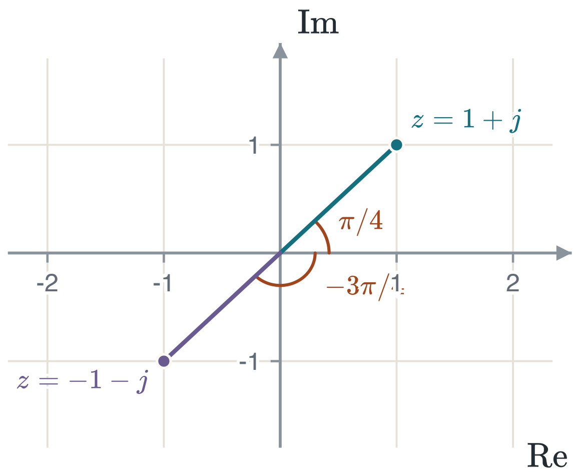

FIGURE 1.39 $1 + j$ and $-1 - j$ give the same $y/x = 1$, but they lie in opposite quadrants. Their angles are $\pi/4$ and $-3\pi/4$.

Continuous-time complex exponentials

DEFINITION

$$x(t) = C\,e^{at}, \qquad C, a \in \mathbb{C}$$

Complex exponentials describe growth, decay and oscillation in one form. The real and imaginary parts of a determine which behaviour occurs.

Case 1: C and a real

If $a < 0$ the signal decays. If $a > 0$ it grows. If $a = 0$ it is the constant C. A larger $|a|$ makes the decay or growth faster.

Each time t advances by $1/|a|$, the exponent changes by 1 in magnitude. So the signal is multiplied by $e \approx 2.72$ when it grows, or divided by e when it decays. For example, $x(t) = e^{-2t}$ reaches e^{-1} where $-2t = -1$, that is at $t = 1/2$. A signal e^{at} that doubles every second has $e^{a\cdot 1} = 2$, so $a = \ln 2 \approx 0.69$.

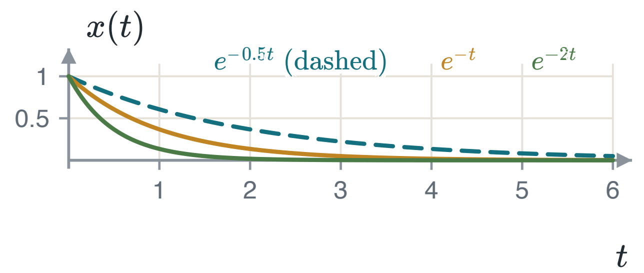

FIGURE 1.40 $a < 0$: decay. A larger $|a|$ decays faster.

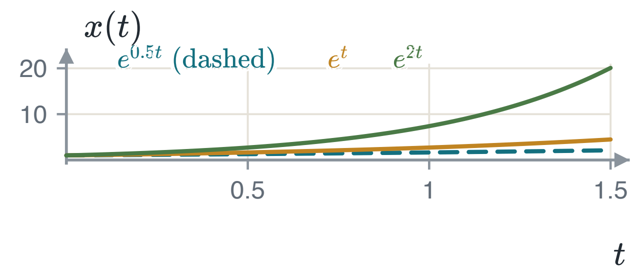

FIGURE 1.41 $a > 0$: growth. Every curve starts at $x(0) = C = 1$.

Case 2: $a = j\omega_0$ purely imaginary

Write $C = Ae^{j\theta}$ in polar form and use Euler's relation:

$$\begin{aligned} x(t) &= Ce^{j\omega_0 t} \\ &= \underbrace{Ae^{j\theta}}_{C}\,e^{j\omega_0 t} \\ &= Ae^{j(\omega_0 t+\theta)} \\ &= A\cos(\omega_0 t + \theta) + jA\sin(\omega_0 t + \theta). \end{aligned}$$

Here $A = |C|$ is the amplitude, ω_0 is the angular frequency in rad/s, and $\theta = \angle C$ is the phase in radians. The amplitude is never negative. For $x(t) = -3e^{j2t}$, write $-3 = 3e^{j\pi}$, so $A = 3$ and $\theta = \pi$.

The modulus is constant: $|x(t)| = A\,|e^{j(\omega_0 t+\theta)}| = A$ for every t, because $|e^{j\phi}| = \sqrt{\cos^2\phi + \sin^2\phi} = 1$. This signal neither grows nor decays. With $\omega_0 = 2\pi f_0$, where f_0 is in hertz, the real part $\cos(2\pi f_0 t)$ is a pure tone of f_0 Hz. Doubling f_0 halves the period and raises the pitch by one octave.

To test periodicity, impose the condition $x(t) = x(t+T)$ and solve for T. Divide both sides by $Ae^{j(\omega_0 t+\theta)}$, which is never zero:

$$\begin{aligned} x(t+T) &= x(t) \\ Ae^{j[\omega_0(t+T)+\theta]} &= Ae^{j(\omega_0 t+\theta)} \\ e^{j\omega_0 T} &= 1 \\ \omega_0 T &= 2\pi k, \qquad k \in \mathbb{Z} \\ T &= \frac{2\pi k}{\omega_0}. \end{aligned}$$

The fourth line holds because $e^{j\phi} = \cos\phi + j\sin\phi$ equals 1 only when $\cos\phi = 1$ and $\sin\phi = 0$, that is when ϕ is a whole number of turns: $e^{j2\pi k} = \cos(2\pi k) + j\sin(2\pi k) = 1 + j0 = 1$.

RESULT

The smallest positive period comes from $k = 1$ when $\omega_0 > 0$, so $T_0 = 2\pi/|\omega_0|$. Every continuous-time complex exponential with $\omega_0 \neq 0$ is periodic. There is no extra condition.

EXAMPLE 1.10

GIVEN $x(t) = e^{j0.5\pi t}$.

FIND The fundamental period.

METHOD Use $T_0 = 2\pi/\omega_0$ because the exponent is purely imaginary and $\omega_0 \neq 0$.

SOLUTION Here $\omega_0 = 0.5\pi$ rad/s, so

$$T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{0.5\pi} = \frac{2}{0.5} = 4\ \text{s}.$$

CHECK $0.5\pi \times 4 = 2\pi$, so the phase advances by one full turn over the calculated period.

Case 3: $a = r + j\omega_0$

Now let a have a real part r and an imaginary part ω_0. Split e^{at} with the law of exponents:

$$\begin{aligned} x(t) &= Ae^{j\theta}e^{(r+j\omega_0)t} \\ &= A\underbrace{e^{rt}}_{\text{envelope}}\underbrace{e^{j(\omega_0 t+\theta)}}_{\text{rotation}} \\ &= Ae^{rt}\cos(\omega_0 t+\theta) + jAe^{rt}\sin(\omega_0 t+\theta). \end{aligned}$$

The real part is $\text{Re}\{x(t)\} = Ae^{rt}\cos(\omega_0 t+\theta)$. The curves $\pm Ae^{rt}$ bound the sinusoid. If $r < 0$, the oscillation is damped. If $r > 0$, it grows. If $r = 0$, it is sustained.

Read r and ω_0 directly from the exponent. For $x(t) = e^{(-1+j4)t}$, $r = \text{Re}\{a\} = -1 < 0$, so the real part $e^{-t}\cos 4t$ decays. A plucked string and a car spring after a bump sound and move like a tone with $r < 0$.

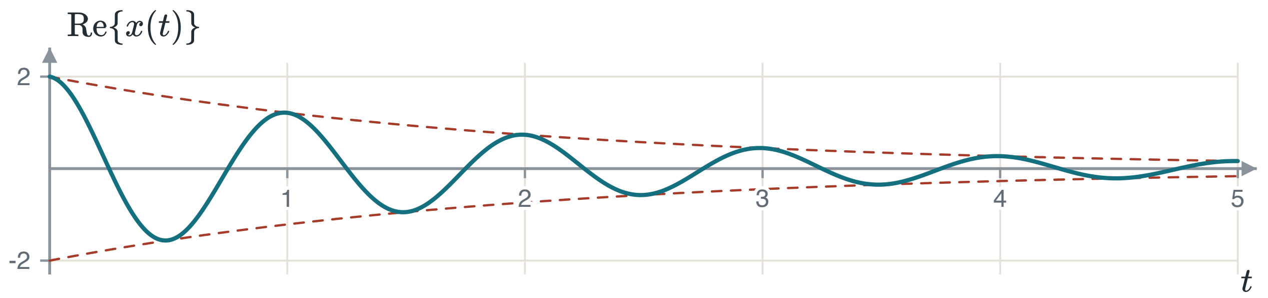

FIGURE 1.42 A damped sinusoid, $\mathrm{Re}\{2e^{-0.5t}e^{j2\pi t}\}$, with $A = 2$, $r = -0.5$ and $\omega_0 = 2\pi$ rad/s. The envelope $\pm 2e^{-0.5t}$ is dashed.

A sum of two exponentials

A sum of two complex exponentials with different frequencies can be written as one exponential times a cosine. Take out the exponential at the average frequency. What is left is two exponentials with opposite frequencies, and Euler's relation turns them into a cosine:

$$\begin{aligned} e^{j\phi} + e^{-j\phi} &= (\cos\phi + j\sin\phi) + (\cos\phi - j\sin\phi) \\ &= 2\cos\phi. \end{aligned}$$

EXAMPLE 1.11

GIVEN $x(t) = e^{j3t} + e^{j5t}$.

FIND $x(t)$ as one exponential times a real signal, and $|x(t)|$.

METHOD Factor out e^{j4t}, because $4 = (3+5)/2$ is the average frequency. Then use $e^{j\phi} + e^{-j\phi} = 2\cos\phi$ and $|e^{j4t}| = 1$.

SOLUTION

$$\begin{aligned} x(t) &= e^{j3t} + e^{j5t} \\ &= e^{j4t}\left(e^{-jt} + e^{jt}\right) \\ &= e^{j4t}\,\underbrace{2\cos t}_{e^{-jt}+e^{jt}}, \\ |x(t)| &= \underbrace{\left|e^{j4t}\right|}_{1}\,|2\cos t| = 2|\cos t|. \end{aligned}$$

CHECK At $t = 0$ both terms equal 1, so $x(0) = 2$, and the formula gives $2e^0\cos 0 = 2$. At $t = \pi/2$, $x = e^{j3\pi/2} + e^{j5\pi/2} = -j + j = 0$, and $2|\cos(\pi/2)| = 0$.

The same steps work for any two frequencies $\omega - d$ and $\omega + d$: $e^{j(\omega-d)t} + e^{j(\omega+d)t} = 2e^{j\omega t}\cos(dt)$, with $|x(t)| = 2|\cos(dt)|$. The magnitude repeats every π/d, so a smaller gap makes it change more slowly. For $x(t) = e^{j2t} + e^{j8t}$, the average is $\omega = 5$ and the half gap is $d = 3$, so $x(t) = 2e^{j5t}\cos 3t$ and $|x(t)| = 2|\cos 3t|$.

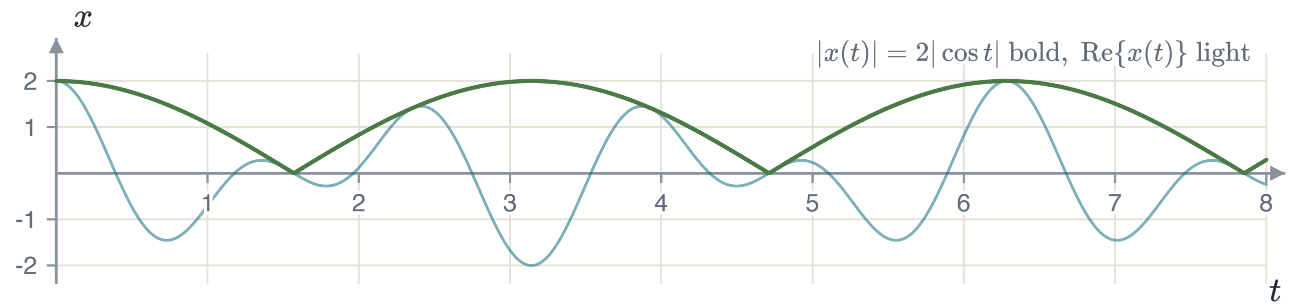

FIGURE 1.43 $x(t) = e^{j3t} + e^{j5t}$. The magnitude $2|\cos t|$ repeats every π. The real part $2\cos t \cos 4t$ oscillates inside it.

Discrete-time complex exponentials

DEFINITION AND POWER FORM

$$x[n] = C\, e^{\beta n}, \qquad C, \beta \in \mathbb{C}$$

$$\alpha = e^{\beta} \quad \Longrightarrow \quad x[n] = Ce^{\beta n} = C\left(e^{\beta}\right)^{n} = C\alpha^{n}$$

Use the power form in discrete time, because the solutions of difference equations have this form.

For real C and α: if $0 < \alpha < 1$ the sequence decreases, and if $\alpha > 1$ it increases. For example, $x[n] = \alpha^n$ with $x[3] = 8$ has $\alpha^3 = 8$, so $\alpha = 2$ and the sequence 2^n grows. For complex $C = |C|e^{j\theta}$ and $\alpha = |\alpha|e^{j\omega_0}$, use the polar forms and the law of exponents:

$$\begin{aligned} x[n] &= C\alpha^n \\ &= \underbrace{|C|e^{j\theta}}_{C}\underbrace{\left(|\alpha|e^{j\omega_0}\right)}_{\alpha}{}^{n} \\ &= |C||\alpha|^n e^{j(\omega_0 n + \theta)} \\ &= |C||\alpha|^n \cos(\omega_0 n + \theta) + j|C||\alpha|^n \sin(\omega_0 n + \theta). \end{aligned}$$

Read the modulus $|\alpha|$ as the envelope. If $|\alpha| = 1$ the oscillation is sustained. If $|\alpha| > 1$ it grows. If $|\alpha| < 1$ it decays. The angle ω_0 sets only the oscillation, in radians per sample. For example, $x[n] = \left(1.1e^{j\pi/3}\right)^n$ grows because $|\alpha| = 1.1 > 1$.

A negative real α is a special case with $\omega_0 = \pi$. For $x[n] = (-0.5)^n$, $n \geq 0$, the samples are $1, -0.5, 0.25, -0.125, \ldots$ The sequence decays because $|\alpha| = 0.5 < 1$, and it changes sign at every step.

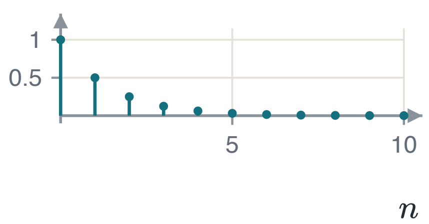

FIGURE 1.44 0.5^n decreases.

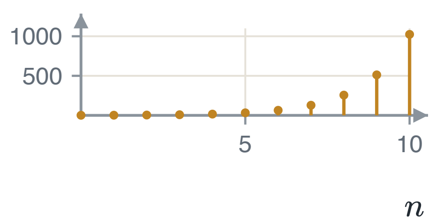

FIGURE 1.45 2^n increases.

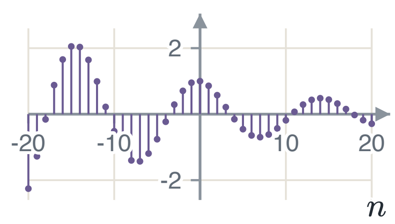

FIGURE 1.46 $0.95^n \cos(0.14\pi n)$: $|\alpha| = 0.95$ decays.

⚠ THE BOUNDARY MOVES

In continuous time the growth-decay boundary is $\mathrm{Re}\{a\} = 0$, the imaginary axis. In discrete time it is $|\alpha| = 1$, the unit circle. The map between them is $\alpha = e^{\beta}$.

Low and high frequencies in discrete time

In discrete time, frequencies that differ by 2π give the same sequence. Split the exponent with the law of exponents:

$$\begin{aligned} e^{j(\omega_0+2\pi)n} &= e^{j\omega_0 n}\, e^{j2\pi n} \\ &= e^{j\omega_0 n} \cdot 1 \\ &= e^{j\omega_0 n}. \end{aligned}$$

The second line holds because $e^{j2\pi n} = \cos(2\pi n) + j\sin(2\pi n) = 1$ for every integer n. So one interval of length 2π holds every distinct sequence. Near $\omega_0 = 0$ or 2π the samples change slowly. Near $\omega_0 = \pi$ they change fastest. At $\omega_0 = \pi$,

$$e^{j\pi n} = \cos(\pi n) + j\underbrace{\sin(\pi n)}_{0} = (-1)^n.$$

This sequence changes sign at every sample. No discrete-time sinusoid changes faster. For example, $\cos(11\pi n/6)$ looks fast but is slow: $11\pi/6 = 2\pi - \pi/6$, so $\cos(11\pi n/6) = \cos(2\pi n - \pi n/6) = \cos(-\pi n/6) = \cos(\pi n/6)$. The last two steps use the 2π-periodicity of the cosine and the fact that it is even.

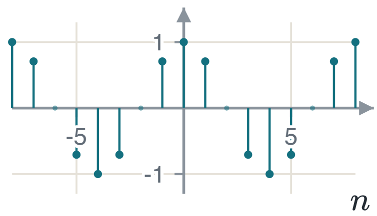

FIGURE 1.47 $\omega_0 = \pi/4$: slow.

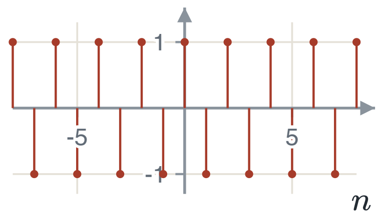

FIGURE 1.48 $\omega_0 = \pi$: $(-1)^n$, the fastest.

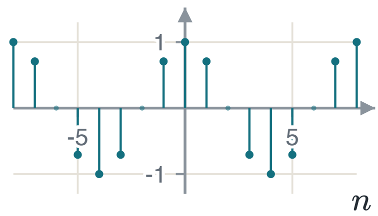

FIGURE 1.49 $\omega_0 = 7\pi/4 = 2\pi - \pi/4$: the same samples as $\pi/4$.

When is a discrete-time exponential periodic?

A discrete-time period must be an integer. Apply $x[n] = x[n+N]$ to $x[n] = Ce^{j\omega_0 n}$ and solve for that integer. As in continuous time, divide by $Ce^{j\omega_0 n}$ and use the condition for $e^{j\phi} = 1$:

$$\begin{aligned} x[n+N] &= x[n] \\ Ce^{j\omega_0(n+N)} &= Ce^{j\omega_0 n} \\ e^{j\omega_0 N} &= 1 \\ \omega_0 N &= 2\pi k, \qquad k \in \mathbb{Z} \\ N &= \frac{2\pi k}{\omega_0}. \end{aligned}$$

PERIODICITY CONDITION

$$\frac{\omega_0}{2\pi} = \frac{k}{N} \in \mathbb{Q}$$

N must be an integer, and so must k. This is possible only when $\omega_0/2\pi$ is rational. If the ratio is irrational, no integer N satisfies the periodicity condition and the sequence is aperiodic.

For example, $x[n] = e^{j2n}$ has $\omega_0/2\pi = 2/2\pi = 1/\pi$, which is irrational. No integer period exists, so the sequence is aperiodic. The values $N = \pi$ and $N = 2\pi$ are not periods, because a period must be an integer. In the same way, $\cos(n/4)$ has $\omega_0/2\pi = 1/(8\pi)$ and is aperiodic.

TABLE 1.2 Continuous-time and discrete-time complex exponentials compared.

	CONTINUOUS TIME, $e^{j\omega_0 t}$	DISCRETE TIME, $e^{j\omega_0 n}$
Periodic?	For every $\omega_0 \neq 0$	Only if $\omega_0/2\pi$ is rational
Distinct frequencies	Each ω_0 gives a different signal	ω_0 and $\omega_0 + 2\pi$ give the same sequence

EXAMPLE 1.12

GIVEN $x[n] = e^{j(3\pi/5)n}$.

FIND The fundamental period N_0.

METHOD Use $N = 2\pi k/\omega_0$ and take the smallest positive integer k that makes N an integer.

SOLUTION Substitute the frequency before choosing k:

$$\begin{aligned} N &= \frac{2\pi k}{\omega_0} \\ &= \frac{2\pi k}{3\pi/5} \\ &= \frac{10}{3}k. \end{aligned}$$

$k = 1$ gives $10/3$ and $k = 2$ gives $20/3$, neither an integer. The smallest positive integer k giving an integer N is $k = 3$, so $N_0 = 10$.

CHECK $\omega_0 N_0 = \frac{3\pi}{5} \times 10 = 6\pi = 2\pi \times 3$. The phase therefore advances by three full turns in ten samples.

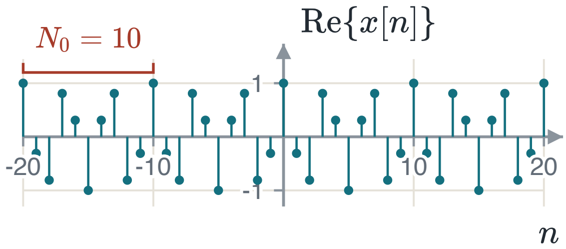

FIGURE 1.50 $\mathrm{Re}\{e^{j3\pi n/5}\}$ repeats every 10 samples.

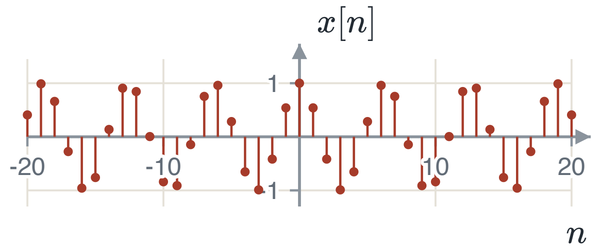

FIGURE 1.51 $\cos(n)$ has $\omega_0 = 1$, so $\omega_0/2\pi = 1/(2\pi)$ is irrational. The samples never repeat exactly. It looks periodic, but it is not.

A sampled sinusoid has its own period

A sequence may take its values from a periodic curve and still have a different period from the curve.

EXAMPLE 1.13

GIVEN $x[n] = \cos(6\pi n/17)$, the values of the curve $\cos(6\pi t/17)$ at integer t.

FIND The period T_0 of the curve, the fundamental period N_0 of the sequence, and the fundamental frequency of the sequence.

METHOD The curve has $T_0 = 2\pi/\omega_0$. The sequence needs the smallest integer $N = 2\pi k/\omega_0$.

SOLUTION With $\omega_0 = 6\pi/17$:

$$T_0 = \frac{2\pi}{6\pi/17} = \frac{17}{3} \approx 5.67,$$
$$N = \frac{2\pi k}{6\pi/17} = \frac{17}{3}k.$$

$k = 1$ and $k = 2$ give $17/3$ and $34/3$, which are not integers. $k = 3$ gives $N_0 = 17 = 3T_0$. The fundamental frequency of the sequence is

$$\frac{2\pi}{N_0} = \frac{2\pi}{17} = \frac{\omega_0}{3}.$$

CHECK One period of the sequence holds 3 cycles of the curve: $\omega_0 N_0 = \frac{6\pi}{17} \cdot 17 = 6\pi = 3 \cdot 2\pi$. So the fundamental frequency is $\omega_0/3$, not ω_0.

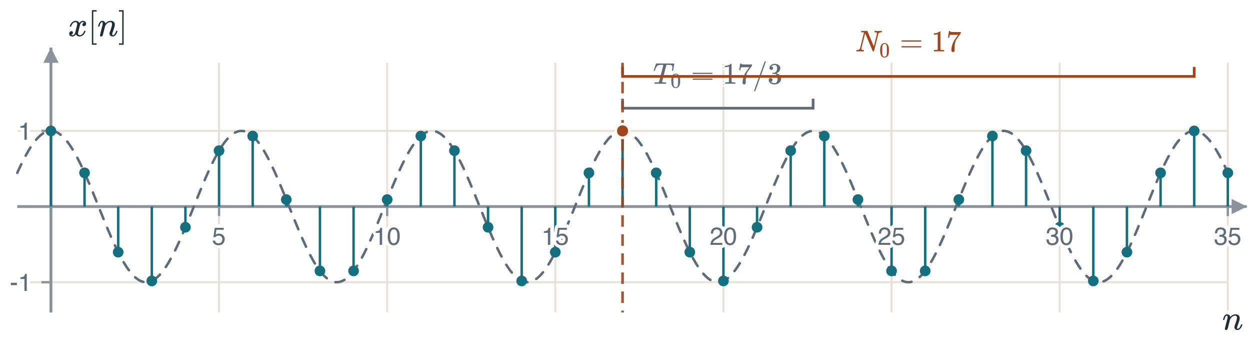

FIGURE 1.52 The dashed curve $\cos(6\pi t/17)$ repeats every $17/3$, but no sample falls at the end of its first or second cycle. After three cycles, at $n = 17$, a sample falls on the start of a cycle again. The brackets mark one cycle of the curve and one period of the sequence, both from $n = 17$.

A second case: $x[n] = \cos(4\pi n/9)$ has $N = 2\pi k/(4\pi/9) = 9k/2$. The smallest integer comes at $k = 2$, so $N_0 = 9$.

Harmonically related exponentials

Chapter 4 builds a periodic signal as a weighted sum of exponentials that share its period. This family has a name. In continuous time, with $\omega_0 = 2\pi/T_0$,

> CONTINUOUS-TIME HARMONIC FAMILY
>
> $$\phi_k(t) = e^{jk\omega_0 t}, \qquad \omega_0 = \frac{2\pi}{T_0}, \qquad k = 0, \pm 1, \pm 2, \dots$$
>
> For $k \neq 0$, ϕ_k has fundamental period $T_0/|k|$. It makes $|k|$ full cycles in T_0, so every member repeats after T_0. The frequencies $k\omega_0$ all differ, so all the members are different.

To check that T_0 is a period of every member, shift by T_0 and use $\omega_0 T_0 = 2\pi$: $\phi_k(t + T_0) = e^{jk\omega_0 t}\, e^{jk\omega_0 T_0} = \phi_k(t)\, e^{j2\pi k} = \phi_k(t)$.

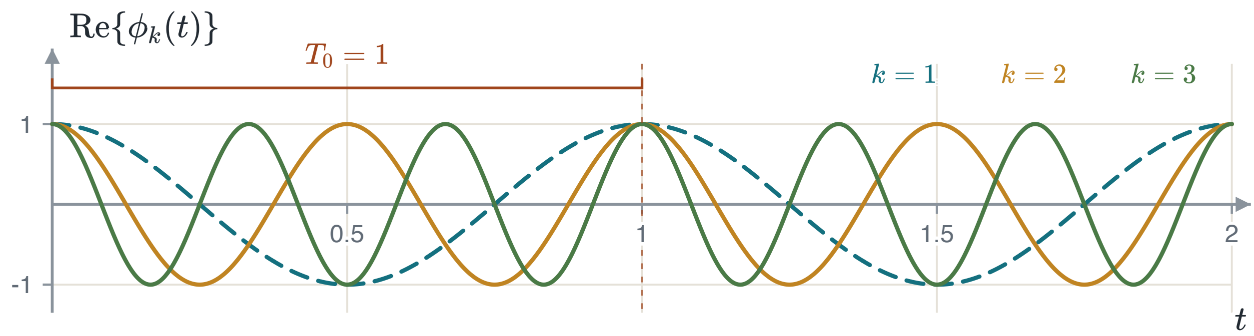

FIGURE 1.53 The harmonics $k = 1$ (dashed), $k = 2$ and $k = 3$ with $T_0 = 1$. The k-th makes k full cycles in T_0, so all of them start together again at $t = T_0$.

In discrete time the family with period N uses the frequencies $k\,2\pi/N$. Adding N to the index adds 2π to the frequency, which changes nothing:

$$\phi_k[n] = e^{jk(2\pi/N)n},$$
$$\phi_{k+N}[n] = e^{jk(2\pi/N)n}\, e^{jN(2\pi/N)n} = \phi_k[n]\,\underbrace{e^{j2\pi n}}_{1} = \phi_k[n].$$

So only N members are different, for example $k = 0, 1, \ldots, N-1$. With $N = 6$, $\phi_{-1}[n] = \phi_{-1+6}[n] = \phi_5[n]$.

Geometric sums

Sums of discrete-time exponentials are geometric sums. Each term is the one before it times the ratio α. To sum N terms, write the sum, multiply it by α, and subtract:

$$\begin{aligned} S_N &= 1 + \alpha + \alpha^2 + \cdots + \alpha^{N-1} \\ \alpha S_N &= \alpha + \alpha^2 + \cdots + \alpha^{N-1} + \alpha^N \\ S_N - \alpha S_N &= 1 - \alpha^N \\ (1-\alpha) S_N &= 1 - \alpha^N \\ S_N &= \frac{1-\alpha^N}{1-\alpha}, \qquad \alpha \neq 1. \end{aligned}$$

The subtraction cancels every middle term. The last step divides by $1 - \alpha$, which is allowed only for $\alpha \neq 1$. For $\alpha = 1$, every term is 1, so $S_N = N$.

GEOMETRIC SUMS

$$\sum_{n=0}^{N-1} \alpha^n = \begin{cases} \dfrac{1-\alpha^N}{1-\alpha}, & \alpha \neq 1 \\ N, & \alpha = 1 \end{cases}$$

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}, \qquad |\alpha| < 1$$

For $|\alpha| < 1$, $|\alpha^N| = |\alpha|^N \to 0$, so the finite sum tends to $1/(1-\alpha)$. For $|\alpha| \geq 1$ the terms do not shrink to zero, and the infinite sum does not converge.

For example, with $\alpha = 1/3$ the ratio satisfies $|1/3| < 1$, so $\sum_{n=0}^{\infty} (1/3)^n = 1/(1 - 1/3) = 1/(2/3) = 3/2$.

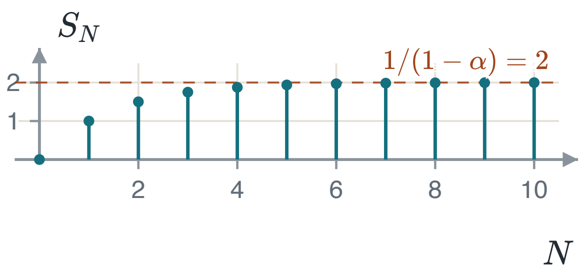

FIGURE 1.54 $\alpha = 0.5$: the partial sums rise towards $1/(1-0.5) = 2$.

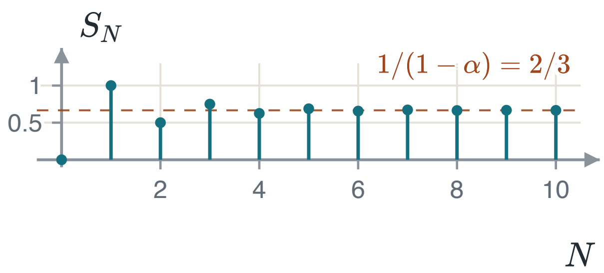

FIGURE 1.55 $\alpha = -0.5$: the partial sums swing above and below $1/(1+0.5) = 2/3$.

Summing one harmonic over a period

The finite geometric sum gives a result that Chapter 4 uses to find Fourier series coefficients. Sum one discrete-time harmonic over one period. Its terms form a geometric sum with ratio $\alpha = e^{jk2\pi/N}$:

$$\sum_{n=0}^{N-1} e^{jk(2\pi/N)n} = \sum_{n=0}^{N-1} \alpha^n, \qquad \alpha = e^{jk2\pi/N}.$$

If k is not a multiple of N, then $k2\pi/N$ is not a whole number of turns, so $\alpha \neq 1$. Use the finite sum, and note that $\alpha^N = e^{jk2\pi} = 1$:

$$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha} = \frac{1-e^{jk2\pi}}{1-\alpha} = \frac{1-1}{1-\alpha} = 0.$$

If k is a multiple of N, then $\alpha = 1$ and all N terms equal 1, so the sum is N.

ONE HARMONIC OVER A PERIOD

$$\sum_{n=0}^{N-1} e^{jk(2\pi/N)n} = \begin{cases} N, & k = 0, \pm N, \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$$

For example, with $N = 4$ and $k = 1$ the terms $e^{j\pi n/2}$ for $n = 0, 1, 2, 3$ are $1,\ j,\ -1,\ -j$, and they add to 0. On the unit circle, the terms of a harmonic with k not a multiple of N are spread evenly, so they balance about the origin.

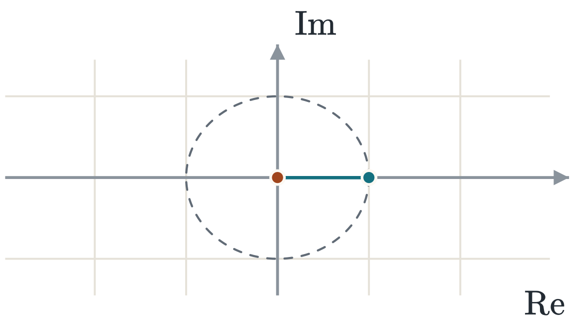

FIGURE 1.56 $N = 6$, $k = 0$: all six terms sit at 1, and the sum is 6.

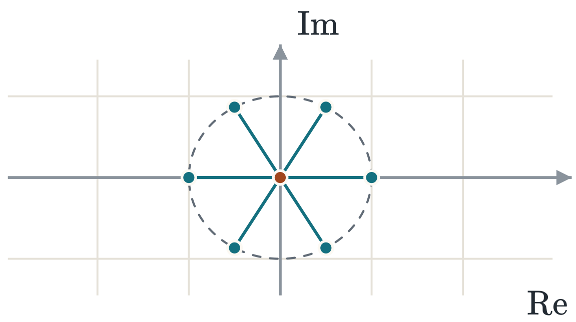

FIGURE 1.57 $N = 6$, $k = 1$: the six terms $e^{j\pi n/3}$ are spread evenly over the circle, and the sum is 0.

1.7 A catalogue of common signals

A small set of named signals appears again and again in later chapters. This section collects them with their formulas. It introduces no new result. Each one is built from the ideas of this chapter.

Building blocks: step, ramp, sign and exponential

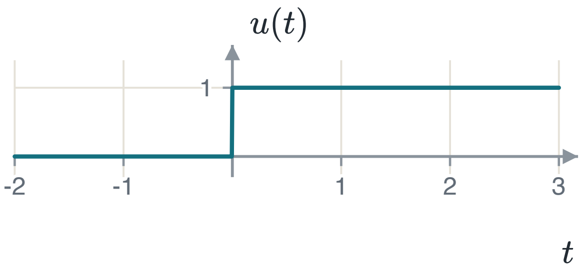

FIGURE 1.58 Unit step: $u(t) = 1$ for $t > 0$ and $u(t) = 0$ for $t < 0$.

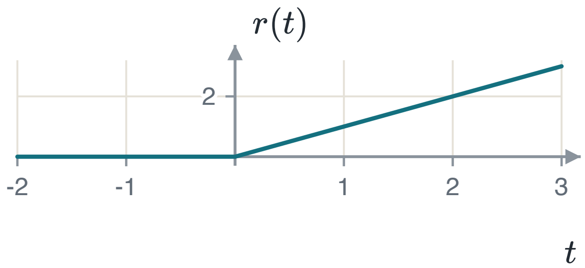

FIGURE 1.59 Unit ramp: $r(t) = t\,u(t)$. Its slope is 1 for $t > 0$, so $\mathrm{d}r/\mathrm{d}t = u(t)$.

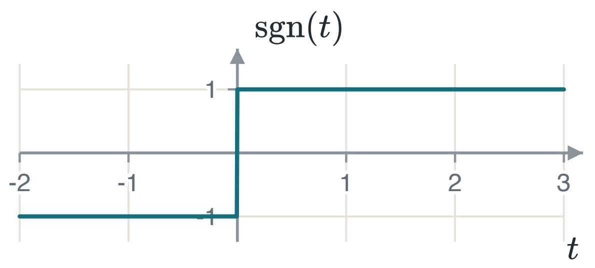

FIGURE 1.60 Sign: $\mathrm{sgn}(t) = 1$ for $t > 0$ and -1 for $t < 0$, so $\mathrm{sgn}(t) = 2u(t) - 1$ for $t \neq 0$.

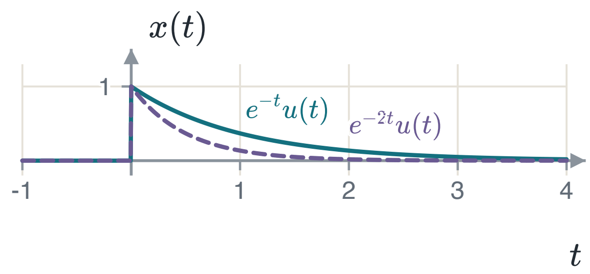

FIGURE 1.61 One-sided exponential: $x(t) = e^{-at}u(t)$ with $a > 0$. Its energy is $1/(2a)$. A larger a decays faster.

Many signals are sums, shifts and products of these four shapes. A voltage V switched on at t_0 is $V\,u(t - t_0)$. The ramp is the running integral of the step, $r(t) = \int_{-\infty}^{t} u(\tau)\,\mathrm{d}\tau$. The step is the running integral of $\delta(t)$. The sign function checks both formulas: for $t > 0$, $2u(t) - 1 = 2 - 1 = 1$, and for $t < 0$, $2u(t) - 1 = 0 - 1 = -1$.

The energy of the one-sided exponential follows from the definition of E_∞. The signal is zero for $t < 0$, so the lower limit is 0:

$$
\begin{aligned}
E_\infty &= \int_0^\infty \left(e^{-at}\right)^2 \mathrm{d}t = \int_0^\infty e^{-2at}\,\mathrm{d}t \\
&= \left[-\frac{e^{-2at}}{2a}\right]_0^\infty = 0 - \left(-\frac{1}{2a}\right) = \frac{1}{2a}, \qquad a > 0.
\end{aligned}
$$

Pulses: rectangle, triangle, sinc and Gaussian

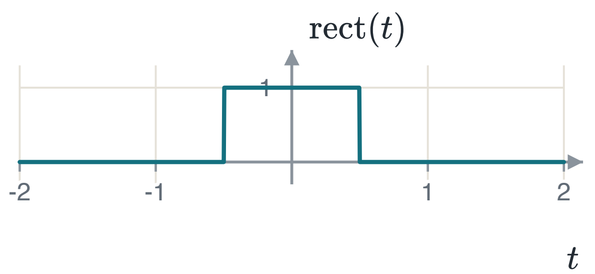

FIGURE 1.62 Rectangle: $\mathrm{rect}(t) = 1$ for $|t| < \frac{1}{2}$ and 0 for $|t| > \frac{1}{2}$. Energy 1.

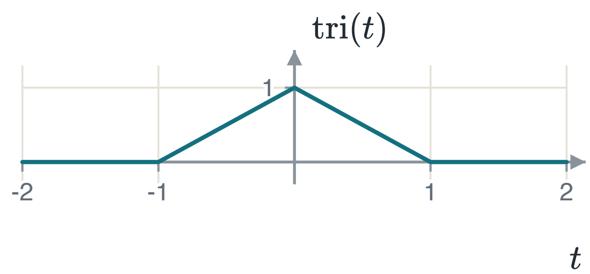

FIGURE 1.63 Triangle: $\mathrm{tri}(t) = 1 - |t|$ for $|t| \leq 1$ and 0 otherwise. Energy $2/3$.

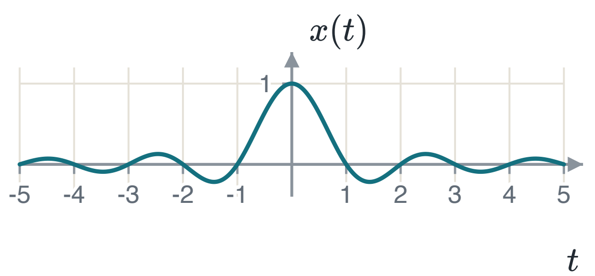

FIGURE 1.64 $x(t) = \mathrm{sinc}(\pi t)$, with the unnormalised $\mathrm{sinc}(\theta) = \sin\theta/\theta$. Peak 1 at $t = 0$; zeros at $t = \pm 1, \pm 2, \ldots$

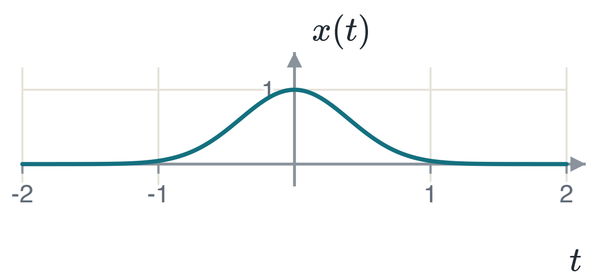

FIGURE 1.65 Gaussian: $x(t) = e^{-\pi t^2}$. It is smooth everywhere and its area is 1.

The sinc is unnormalised here: $\mathrm{sinc}(\theta) = \sin\theta/\theta$, so $\mathrm{sinc}(\pi t) = \sin(\pi t)/(\pi t)$. Its zeros are where $\sin(\pi t) = 0$ with $t \neq 0$, that is at the non-zero integers. At $t = 0$ the limit $\sin\theta/\theta \to 1$ gives the peak 1 .

The rectangle has energy $\int_{-1/2}^{1/2} 1^2\,\mathrm{d}t = 1$. The triangle energy shows how to use symmetry in an energy integral. The integrand is even, so integrate over $[0,1]$ and double. Then substitute $s = 1 - t$, $\mathrm{d}s = -\mathrm{d}t$, which maps $t = 0 \mapsto s = 1$ and $t = 1 \mapsto s = 0$:

$$\begin{aligned} E_\infty &= \int_{-1}^{1} (1 - |t|)^2\,\mathrm{d}t = 2\int_0^1 (1-t)^2\,\mathrm{d}t \\ &= 2\int_1^0 s^2\,(-\mathrm{d}s) = 2\int_0^1 s^2\,\mathrm{d}s = 2\left[\frac{s^3}{3}\right]_0^1 = \frac{2}{3}. \end{aligned}$$

A pulse can shape the loudness of a tone: $p(t - t_0)\cos(2\pi \cdot 440\,t)$. The jumps of the rectangle are heard as clicks, while the Gaussian starts and ends softly. The pulses return later. The triangle is the rectangle convolved with itself (Chapter 3). The rectangle and the sinc are a transform pair (Chapter 5).

Periodic waveforms: sine, square, triangle and sawtooth

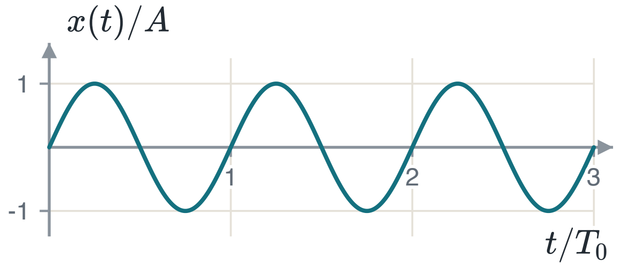

FIGURE 1.66 Sine: $x(t) = A\sin(2\pi t/T_0)$. Average power $A^2/2$.

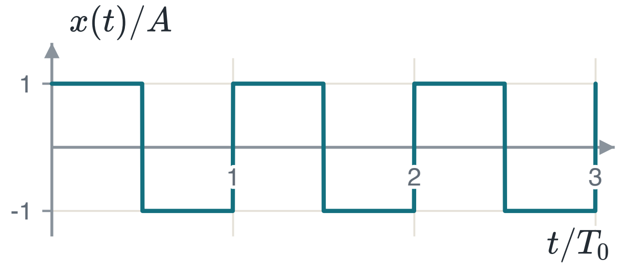

FIGURE 1.67 Square: $x(t) = A$ for $0 \le t < T_0/2$ and $-A$ for $T_0/2 \le t < T_0$. Average power A^2.

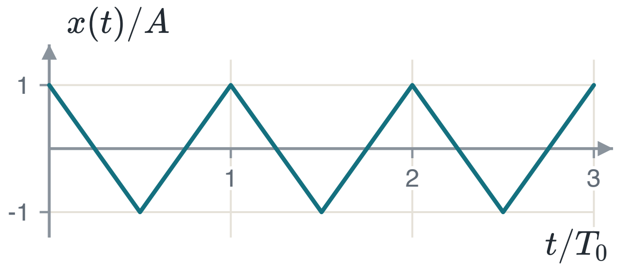

FIGURE 1.68 Triangle: $x(t) = A\,(1 - 4|t|/T_0)$ for $|t| \le T_0/2$. Average power $A^2/3$.

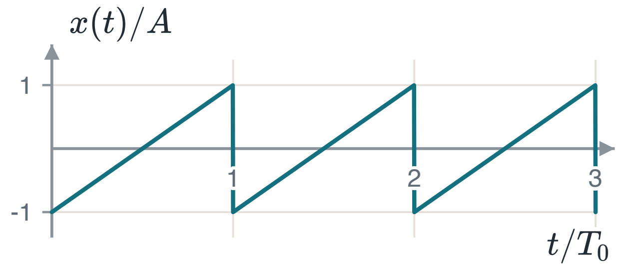

FIGURE 1.69 Sawtooth: $x(t) = A\,(2t/T_0 - 1)$ for $0 \le t < T_0$. Average power $A^2/3$.

For a periodic signal, the average over all time equals the average over one period, $P_\infty = \frac{1}{T_0}\int_0^{T_0} |x(t)|^2\,\mathrm{d}t$, because every period adds the same energy. The square wave has $|x(t)|^2 = A^2$ at every t, so $P_\infty = A^2$. For the sine, use $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$; the cosine term integrates to zero over whole periods:

$$P_\infty = \frac{1}{T_0}\int_0^{T_0} A^2 \sin^2\!\left(\frac{2\pi t}{T_0}\right)\mathrm{d}t = \frac{A^2}{T_0}\int_0^{T_0} \frac{1 - \cos(4\pi t/T_0)}{2}\,\mathrm{d}t = \frac{A^2}{T_0}\cdot\frac{T_0}{2} = \frac{A^2}{2}.$$

For the sawtooth, substitute $s = 2t/T_0 - 1$, $\mathrm{d}t = \frac{T_0}{2}\mathrm{d}s$, which maps $t = 0 \mapsto s = -1$ and $t = T_0 \mapsto s = 1$:

$$P_\infty = \frac{1}{T_0}\int_0^{T_0} A^2\left(\frac{2t}{T_0} - 1\right)^2 \mathrm{d}t = \frac{A^2}{T_0}\cdot\frac{T_0}{2}\int_{-1}^{1} s^2\,\mathrm{d}s$$
$$= \frac{A^2}{2}\left[\frac{s^3}{3}\right]_{-1}^{1} = \frac{A^2}{2}\cdot\frac{2}{3} = \frac{A^2}{3}.$$

For the triangle, average over the period $[-T_0/2, T_0/2]$. The integrand is even, so integrate over $[0, T_0/2]$ and double. Then substitute $s = 1 - 4t/T_0$, $\mathrm{d}t = -\frac{T_0}{4}\mathrm{d}s$, which maps $t = 0 \mapsto s = 1$ and $t = T_0/2 \mapsto s = -1$:

$$P_\infty = \frac{2}{T_0}\int_0^{T_0/2} A^2\left(1 - \frac{4t}{T_0}\right)^2 \mathrm{d}t = \frac{2A^2}{T_0}\int_1^{-1} s^2\left(-\frac{T_0}{4}\right)\mathrm{d}s$$
$$= \frac{A^2}{2}\int_{-1}^{1} s^2\,\mathrm{d}s = \frac{A^2}{2}\cdot\frac{2}{3} = \frac{A^2}{3}.$$

Played with the same period, $T_0 = 1/220$ s, all four have the same pitch. The shape changes only the colour of the sound, which Chapter 4 explains with harmonics. The triangle and the sawtooth have the same average power, $A^2/3$, and still sound different. Power does not fix the shape.

Signals for the ear: chirp, beats, AM and FM

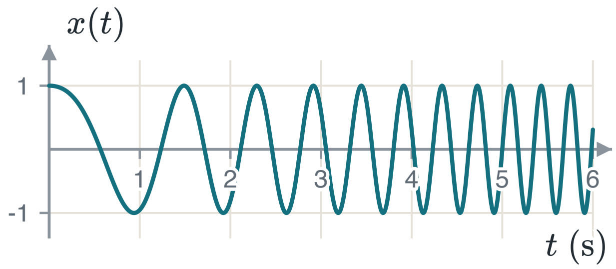

FIGURE 1.70 Chirp: $x(t) = \cos\left(2\pi(f_0 t + \frac{k}{2}t^2)\right)$. Its frequency $f_0 + kt$ rises with time. Drawn with $f_0 = 0.3$ Hz, $k = 0.5$ Hz/s.

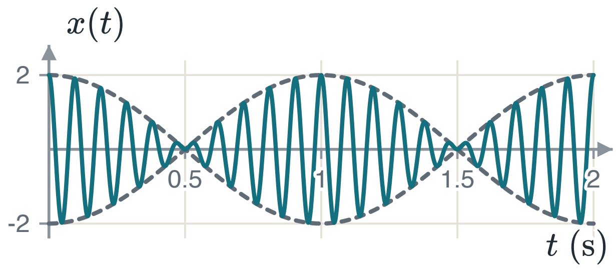

FIGURE 1.71 Beats: $\cos(2\pi f_1 t) + \cos(2\pi f_2 t)$, drawn with $f_1 = 10$ Hz, $f_2 = 11$ Hz. The dashed envelope is $\pm 2\cos(\pi(f_1 - f_2)t)$.

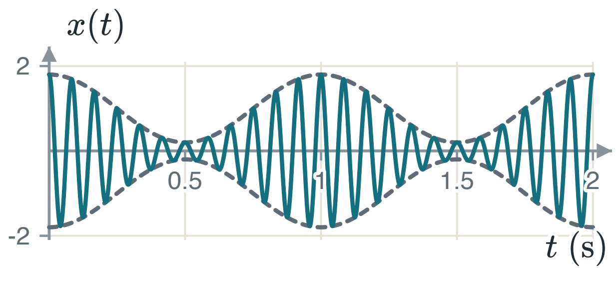

FIGURE 1.72 AM: $x(t) = \left(1 + m\cos(2\pi f_m t)\right)\cos(2\pi f_c t)$ with $0 < m \le 1$. The dashed envelope is $\pm\left(1 + m\cos(2\pi f_m t)\right)$. Drawn with $f_c = 12$ Hz, $f_m = 1$ Hz, $m = 0.8$.

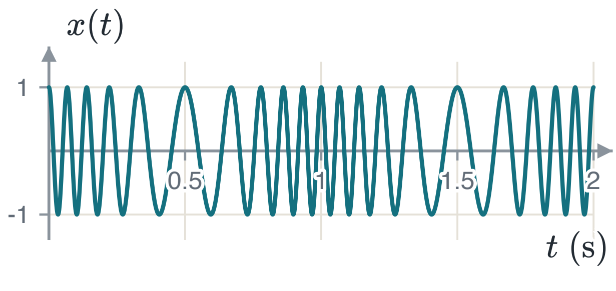

FIGURE 1.73 FM: $x(t) = \cos\left(2\pi f_c t + \beta\sin(2\pi f_m t)\right)$. Drawn with $f_c = 10$ Hz, $f_m = 1$ Hz, $\beta = 5$.

Beats come from the sum-to-product identity $\cos\theta_1 + \cos\theta_2 = 2\cos\left(\frac{\theta_1-\theta_2}{2}\right)\cos\left(\frac{\theta_1+\theta_2}{2}\right)$. With $\theta_i = 2\pi f_i t$, the half-difference is $\pi(f_1 - f_2)t$ and the half-sum is $\pi(f_1+f_2)t$:

$$\cos(2\pi f_1 t) + \cos(2\pi f_2 t) = 2\cos\big(\pi(f_1-f_2)t\big)\cos\big(\pi(f_1+f_2)t\big).$$

The slow factor sets the loudness. The loudness follows $|2\cos(\pi(f_1-f_2)t)|$, which has period $1/|f_1-f_2|$, so it peaks $|f_1-f_2|$ times a second. Two strings a few hertz apart beat. A musician turns the peg until the beats slow down and stop, which means $f_1 = f_2$.

The frequency of a signal $\cos\phi(t)$ at time t, in hertz, is $\frac{1}{2\pi}\frac{\mathrm{d}\phi}{\mathrm{d}t}$. Differentiate the phase of the chirp and of FM:

$$\begin{aligned} \text{chirp:}\quad & \frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}t}\left[2\pi\left(f_0 t + \frac{k}{2}t^2\right)\right] = f_0 + kt, \\ \text{FM:}\quad & \frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}t}\left[2\pi f_c t + \beta\sin(2\pi f_m t)\right] = f_c + \beta f_m\cos(2\pi f_m t). \end{aligned}$$

In FM the amplitude stays fixed and the frequency swings about f_c. A chirp sweeps its frequency, as a bird call or a radar pulse does. Beats and AM keep the tone and move its loudness.

Random signals: noise, a noisy tone, a random walk and a telegraph signal

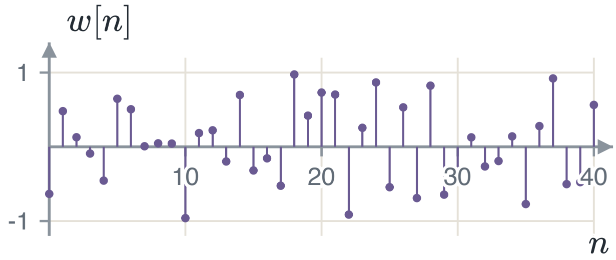

FIGURE 1.74 White noise $w[n]$: the samples are independent, with zero mean and the same variance. No sample predicts the next one.

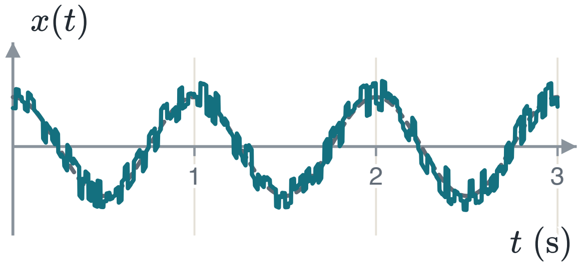

FIGURE 1.75 Signal plus noise: $x(t) = \cos(2\pi f_0 t) + \sigma\, w(t)$. The dashed curve is the tone alone.

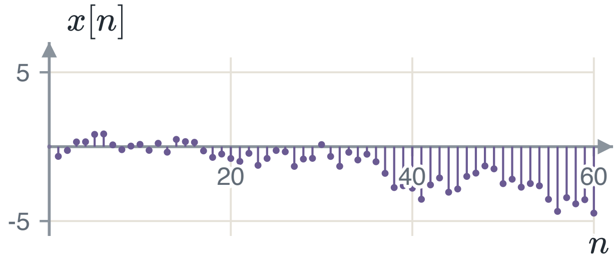

FIGURE 1.76 Random walk: $x[n] = x[n-1] + w[n]$ with $x[0] = 0$. Each step is random, but the sum wanders far from zero.

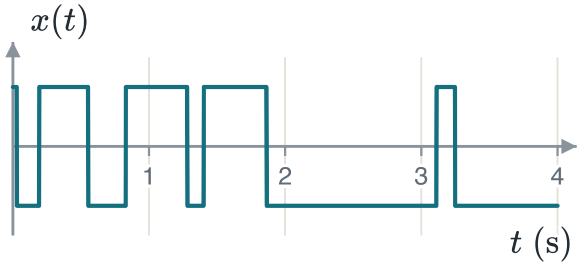

FIGURE 1.77 Random telegraph: $x(t)$ is $+1$ or -1 and flips sign at random times. Only the times of the flips are random.

A random signal has a rule for how it is made, not a formula for each value. Run the rule again and the waveform changes, but its average behaviour does not. A measured signal almost always contains some noise. Heard as sound, white noise is a hiss, the random walk is a low rumble because it changes slowly, and the telegraph signal crackles.

1.8 Summary

1. E_∞ and P_∞ are limits. Finite energy forces zero average power. Finite non-zero average power forces infinite energy. Unbounded growth gives neither.
2. To build $x(at-b)$, shift by b first, then scale by a. The other order gives $x(at-ab)$.
3. T_0 and N_0 are the smallest positive periods, and $\omega_0 = 2\pi/T_0 = 2\pi/N_0$. A sum of periodic signals has the least common multiple of their periods as a period; in continuous time it exists only if the ratio of the periods is rational.
4. Every signal splits in one way into an even and an odd part: $\mathcal{E}\mathrm{v}\{x(t)\} = \frac{1}{2}[x(t)+x(-t)]$ and $\mathcal{O}\mathrm{dd}\{x(t)\} = \frac{1}{2}[x(t)-x(-t)]$. The odd part is 0 at $t=0$, and an odd signal integrates to 0 over $[-a,a]$.
5. $\delta[n] = u[n]-u[n-1]$ and $u[n] = \sum_{k=-\infty}^{n}\delta[k]$: a first difference and a running sum.
6. $\delta[n]$ is an ordinary sequence. $\delta(t)$ is a distribution, defined by sifting. It scales as $\delta(at) = \delta(t)/|a|$.
7. Every sequence is a sum of weighted, shifted impulses: $x[n] = \sum_k x[k]\,\delta[n-k]$.
8. Sifting, $\int x(t)\,\delta(t-t_0)\,\mathrm{d}t = x(t_0)$, gives a **number**. Sampling, $x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$, gives a **signal**.
9. In $e^{(r+j\omega_0)t}$, r sets the envelope e^{rt}: growth for $r>0$, decay for $r<0$. ω_0 sets the oscillation. In discrete time $|\alpha|$ sets the envelope, with the boundary at $|\alpha|=1$.
10. A discrete-time exponential $e^{j\omega_0 n}$ is periodic if and only if $\omega_0/2\pi$ is rational. A continuous-time one with $\omega_0 \neq 0$ always is.
11. $e^{j\omega_0 n}$ and $e^{j(\omega_0+2\pi)n}$ are the same sequence, so discrete-time frequencies repeat every 2π. The fastest sequence is $e^{j\pi n} = (-1)^n$.
12. $\sum_{n=0}^{N-1}\alpha^n = (1-\alpha^N)/(1-\alpha)$ for $\alpha \neq 1$, and one harmonic summed over a period gives N when k is a multiple of N and 0 otherwise.

Exercises

1.1 Classify $x(t) = e^{-3t}u(t)$ as an energy signal, a power signal, or neither. Give the value of whichever quantity is finite.
Answer: Energy signal, $E_\infty = 1/6$.

1.2 For $x[n]$ non-zero only on $-1 \le n \le 1$ with values $1, 2, 1$, sketch $x[n+4]$ and $x[n-5]$ and give the support of each.
Answer: Supports $-5 \le n \le -3$ and $4 \le n \le 6$.

1.3 A signal $x(t)$ is non-zero only on $[-1,5]$. On what interval is $x(2t+3)$ non-zero?
Answer: $[-2,1]$.

1.4 Find the fundamental period of $x[n] = \cos(4\pi n/7)$, or show that it is aperiodic.
Answer: $N_0 = 7$.

1.5 Use symmetry to evaluate $\int_{-2}^{2}\left(t^3 + t\cos t + 1\right)\mathrm{d}t$ without integrating the first two terms.
Answer: 4. The first two terms are odd.

1.6 Evaluate $\int_{-\infty}^{\infty}(t^2+1)\,\delta(t-2)\,\mathrm{d}t$ and $\sum_{n=-\infty}^{\infty} 2^{-n}\,\delta[n-3]$.
Answer: 5 and $1/8$.

1.7 Find the fundamental period of $x(t) = \cos(\pi t/2) + \sin(\pi t/3)$.
Answer: $T_0 = 12$.

1.8 Write $x(t) = e^{jt} + e^{j7t}$ as one exponential times a cosine, and give $|x(t)|$.
Answer: $x(t) = 2e^{j4t} \cos 3t$ and $|x(t)| = 2|\cos 3t|$.

CHAPTER

2

Systems and their properties

A system turns an input signal into an output signal. This chapter describes a system only through its input and its output. It gives tests for six properties: memory, invertibility, causality, stability, time invariance and linearity. For each property it shows how to prove it for every input, and how to disprove it with one counterexample.

2.1 The input–output abstraction

A system is a rule that turns an input signal into an output signal. The rule is deterministic: the same input always gives the same output.

Write the rule as an operator, $y = S\{x\}$. The operator acts on the whole input signal, not on one value of it. So the output at one time may depend on input values at other times. This form lets us ask which input times affect an output, and how the system responds to a time shift.

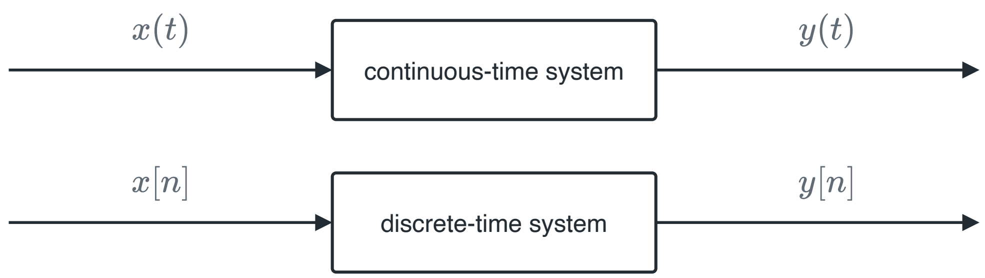

FIGURE 2.1 The same operator description applies in continuous time and in discrete time.

> EQUAL SYSTEMS
>
> Two systems are equal when they give the same output for every input. How they are built does not matter. Let S_1 double its input with an amplifier, and let S_2 add its input to itself. Then $S_1\{x\} = 2x$ and $S_2\{x\} = x + x = 2x$ for every input x, so S_1 and S_2 are the same system.

One equation, many systems

Different physical systems can share one input–output equation. Two examples show this.

An RC circuit. A source voltage $v_s(t)$ drives a resistor R in series with a capacitor C. The output is the capacitor voltage $v_C(t)$. Kirchhoff's voltage law around the loop gives $v_s = Ri + v_C$, where i is the loop current. The capacitor current is $i = C\,\mathrm{d}v_C/\mathrm{d}t$. Substitute it:

$$\begin{aligned} v_s(t) &= R\,i(t) + v_C(t) \\ &= RC\,\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} + v_C(t). \end{aligned}$$

Divide both sides by RC and exchange the two sides:

$$\frac{\mathrm{d}v_C(t)}{\mathrm{d}t} + \frac{1}{RC}\,v_C(t) = \frac{1}{RC}\,v_s(t).$$

A car. A force $f(t)$ pushes a car of mass m. Friction opposes the motion with the force $\rho\,v(t)$, where v is the speed. Newton's second law gives $m\,\mathrm{d}v/\mathrm{d}t = f - \rho\,v$. Move the friction term to the left and divide by m:

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} + \frac{\rho}{m}\,v(t) = \frac{1}{m}\,f(t).$$

ONE FORM

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} + a\,y(t) = b\,x(t)$$

$$y[n] + a\,y[n-1] = b\,x[n]$$

The circuit has the first form with $a = b = 1/RC$. The car has it with $a = \rho/m$ and $b = 1/m$. The second line is the discrete-time form. One method of analysis then serves every system of the same form; Section 3.5 gives it.

A savings account has the discrete-time form. With 1 % interest a month, the balance is $y[n] = 1.01\,y[n-1] + x[n]$, where $x[n]$ is the deposit in month n. Move $1.01\,y[n-1]$ to the left: $y[n] - 1.01\,y[n-1] = x[n]$. So $a = -1.01$ and $b = 1$.

With equal time constants $\tau = RC = m/\rho$, the circuit and the car respond in the same way. Switch on a constant input at $t = 0$, with the system at rest. In both systems the output, divided by its final value, is $1 - e^{-t/\tau}$ for $t > 0$. For the circuit with $v_s(t) = 1$ for $t > 0$, substitute $v_C = 1 - e^{-t/\tau}$ into the left side of the equation:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\big(1 - e^{-t/\tau}\big) + \frac{1}{\tau}\big(1 - e^{-t/\tau}\big) &= \frac{1}{\tau}\,e^{-t/\tau} + \frac{1}{\tau} - \frac{1}{\tau}\,e^{-t/\tau} \\ &= \frac{1}{\tau} = \frac{1}{RC}\,v_s(t). \end{aligned}$$

The left side equals the right side, so the curve solves the equation. It also starts at $v_C(0) = 1 - 1 = 0$, as a system at rest must. Section 3.5 shows how to find such a solution.

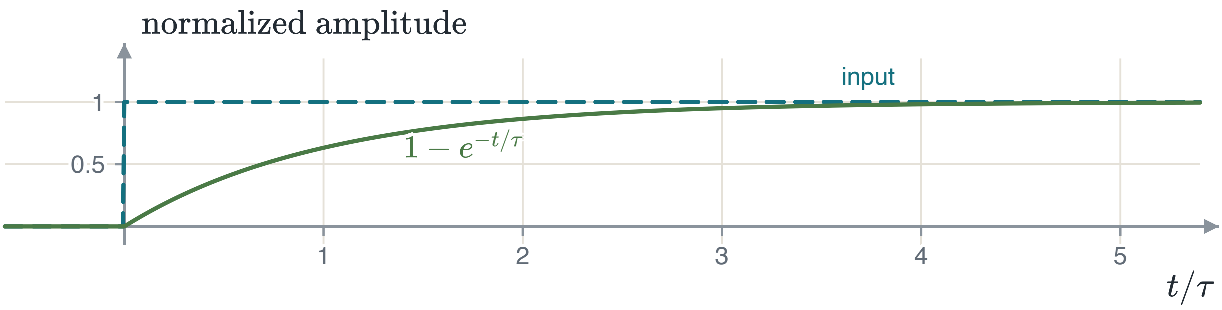

FIGURE 2.2 A constant input switched on at rest. The capacitor voltage and the speed of the car both follow $1 - e^{-t/\tau}$.

> ⚠ A MODEL HAS LIMITS
>
> Ohm's law and linear friction are idealizations. An analysis holds only while the signals stay in the range where the model is accurate. An amplifier is linear only until it clips. A circuit that warms up is only close to time invariant.

Connecting systems

Larger systems are built by connecting smaller ones. Three connections occur again and again.

- **Series**, or cascade: the output of S_1 is the input of S_2.
- **Parallel**: both systems get the same input, and their outputs add.
- **Feedback**: the output of S_1 passes through S_2 and is added to the input of S_1.

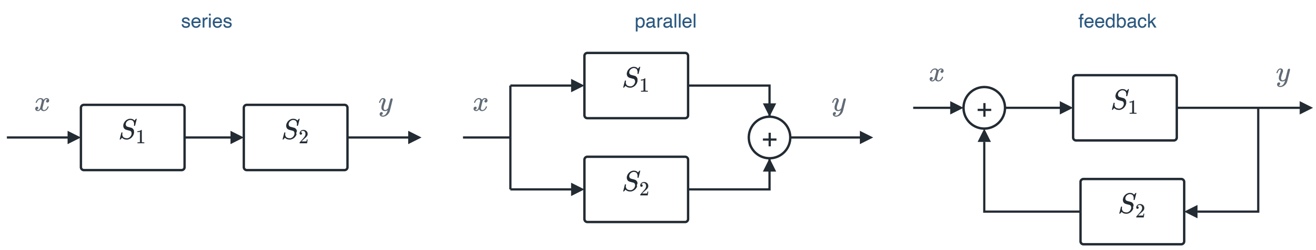

FIGURE 2.3 Series, parallel and feedback connections of two systems. The circle adds the signals that enter it.

In the feedback connection, let S_1 pass its input unchanged and let S_2 be a one-sample delay. The signal that enters S_1 is then $x[n] + y[n-1]$, and S_1 passes it on:

$$y[n] = x[n] + y[n-1].$$

This system is the accumulator. Section 2.2 shows that it remembers every earlier input.

The order of a series connection can change the result. Squaring and then doubling gives $2x^2$. Doubling and then squaring gives $(2x)^2 = 4x^2$.

EXAMPLE 2.1

GIVEN S_1: $y[n] = x[n-1]$, followed in series by S_2: $y[n] = 2\,x[n]$.

FIND The rule of the whole connection.

METHOD Name the signal between the two systems. Write each system rule for it, then substitute one rule into the other.

SOLUTION Call the output of S_1 $w[n]$. Then $w[n] = x[n-1]$. The system S_2 doubles its own input, which is $w[n]$:

$$\begin{aligned} y[n] &= 2\,w[n] \\ &= 2\,x[n-1]. \end{aligned}$$

The connection delays the input by one sample and doubles it.

CHECK Take $x[n] = \delta[n]$. Then $w[n] = \delta[n-1]$ and $y[n] = 2\,\delta[n-1]$. The rule $y[n] = 2\,x[n-1]$ gives $2\,\delta[n-1]$ for the same input.

2.2 The six properties

Each property is a statement about every input. To prove a property, give an argument that holds for every input. To disprove it, give one explicit input for which it fails. One example that works cannot prove a property.

Memory

CRITERION

A system is **memoryless** if the output at time t, or at n, depends only on the input at that same time.

TABLE 2.1 Memory: verdicts for example systems.

SYSTEM	VERDICT	REASON
$y(t) = \left[2x(t) - x^2(t)\right]^2$	memoryless	Only $x(t)$ appears. There is no $x(t+1)$ or $x(t-2)$ term.
$y[n] = x[n]$	memoryless	The identity system.
$y[n] = x[n-1]$	has memory	The output at n uses the sample at $n-1$.
$y(t) = x(t/2)$	has memory	At $t = 2$ the output uses $x(1)$, an input at a different time.
$y[n] = x[n] + y[n-1]$	has memory	See the derivation below.

The last system is the accumulator from Section 2.1. Assume initial rest: the output is zero until the input starts. Then substitute the feedback relation into itself, again and again:

$$\begin{aligned} y[n] &= x[n] + y[n-1] \\ &= x[n] + x[n-1] + y[n-2] \\ &= x[n] + x[n-1] + x[n-2] + y[n-3] \\ &\vdots \\ &= x[n] + x[n-1] + x[n-2] + \cdots \\ &= \sum_{k=0}^{\infty} x[n-k]. \end{aligned}$$

The result uses $x[n-k]$ for every $k \geq 0$, so the output depends on the whole input history. Output feedback gives the system memory. For the impulse input $x[n] = \delta[n]$, the term $\delta[n-k]$ is 1 only at $k = n$, and such a $k \geq 0$ exists only for $n \geq 0$:

$$y[n] = \sum_{k=0}^{\infty} \delta[n-k] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases} = u[n].$$

The impulse enters once, and the feedback keeps it. So $y[3] = 1$.

CIRCUIT EXAMPLES

A resistor, $v(t) = R\,i(t)$, is memoryless: the voltage at t uses only the current at t. A capacitor is not memoryless, because $v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)\,\mathrm{d}\tau$ uses the whole current history.

Invertibility

CRITERION

A system is **invertible** if distinct inputs always produce distinct outputs. The map from inputs to outputs must be one-to-one.

To prove invertibility, find a formula that recovers every input from its output. To disprove invertibility, find two distinct inputs that give the same output.

EXAMPLE 2.2

GIVEN $y(t) = \big[\cos(t) + 2\big]x(t)$.

FIND Is the system invertible?

METHOD Solve the system rule for $x(t)$, and show that the divisor is never zero.

SOLUTION First bound the gain. Add 2 to each part of the bound on the cosine:

$$\begin{aligned} -1 \leq \cos(t) &\leq 1 \\ 1 \leq \cos(t) + 2 &\leq 3. \end{aligned}$$

The gain is never zero. Now divide the system equation by it:

$$y(t) = \big[\cos(t) + 2\big]x(t) \quad \Longrightarrow \quad x(t) = \frac{y(t)}{\cos(t) + 2}.$$

Every input is recovered from its output, so the system is invertible.

CHECK Substitute the formula back: $\big[\cos t + 2\big]\dfrac{y(t)}{\cos t + 2} = y(t)$. With the gain $\cos(t)$ alone, the system would lose the input at every zero of the cosine.

EXAMPLE 2.3

GIVEN $y(t) = x^2(t)$.

FIND Is the system invertible?

METHOD Look for two distinct inputs that squaring maps to the same output.

SOLUTION Take $x_1(t) = 1$ and $x_2(t) = -1$ for all t. Then

$$S\{x_1\} = 1^2 = 1,$$
$$S\{x_2\} = (-1)^2 = 1.$$

The inputs are different, but their outputs are equal. The system is **not** invertible.

CHECK One counterexample is enough. The sign of the input is lost and cannot be recovered from y alone.

> ⊗ NOT AN INVERSION FORMULA
>
> Writing $x(t) = \sqrt{y(t)}$ does not invert $y = x^2$. It keeps only one of the two possible inputs. An inversion formula must return the actual input, for every input.

Inverse systems

An invertible system S has an **inverse system**. Placed in series after S, the inverse returns the input: its output w equals x for every input x.

The accumulator $y[n] = \sum_{k=-\infty}^{n} x[k]$ has the first difference $w[n] = y[n] - y[n-1]$ as its inverse. To show this, split the last term off the first sum. Every other term then appears in both sums and cancels:

$$\begin{aligned} w[n] &= y[n] - y[n-1] \\ &= \sum_{k=-\infty}^{n} x[k] - \sum_{k=-\infty}^{n-1} x[k] \\ &= \left(\sum_{k=-\infty}^{n-1} x[k] + x[n] \right) - \sum_{k=-\infty}^{n-1} x[k] \\ &= x[n]. \end{aligned}$$

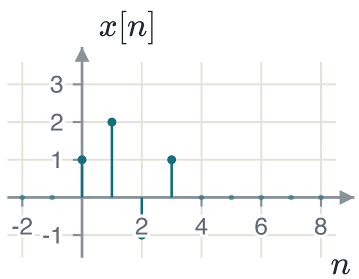

FIGURE 2.4 The input $x[n]$.

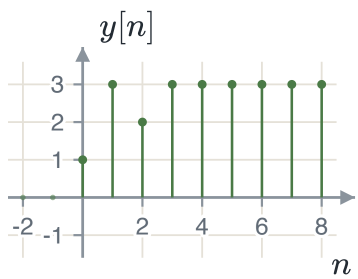

FIGURE 2.5 The accumulator output $y[n]$.

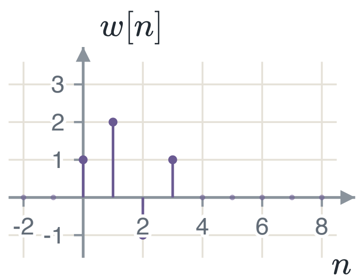

FIGURE 2.6 The first difference: $w[n] = x[n]$.

A lossless encoder must be invertible. The decoder is its inverse system and recovers the message exactly.

EXAMPLE 2.4

GIVEN $y(t) = 3\,x(t-2)$.

FIND The inverse system.

METHOD The system delays the input by 2 and then multiplies it by 3. Undo the two operations in reverse order: divide by 3, then advance by 2.

SOLUTION The inverse is

$$w(t) = \tfrac{1}{3}\,y(t+2).$$

CHECK Substitute the system rule, evaluated at $t+2$:

$$\begin{aligned} w(t) &= \tfrac{1}{3}\,y(t+2) \\ &= \tfrac{1}{3}\cdot 3\,x\big((t+2)-2\big) \\ &= x(t). \end{aligned}$$

The cascade returns the input. The candidates $\frac{1}{3}\,y(t-2)$ and $3\,y(t+2)$ return $x(t-4)$ and $9\,x(t)$ instead.

Causality

CRITERION

A system is **causal** if the output at time t, or at n, depends only on inputs at times up to t: the present and the past.

TABLE 2.2 Causality: verdicts for example systems.

SYSTEM	VERDICT	REASON
$y[n] = x[n-1]$	causal	Uses only a past sample.
$y[n] = x[n] + x[n+1]$	not causal	$x[n+1]$ is a future sample.
$y(t) = \int_{-\infty}^{t} x(\tau)\,\mathrm{d}\tau$	causal	The upper limit is t, so only $\tau \le t$ contributes.
$y[n] = x[-n]$	not causal	$y[-1] = x[-(-1)] = x[1]$. The output at $n=-1$ needs a future input.
$y(t) = x(2t)$	not causal	At $t=1$ the output uses $x(2)$, a future value.
$y(t) = x(t)\cos(t+1)$	causal	Uses only $x(t)$. The factor $\cos(t+1)$ is a known function of t, fixed in advance. It is not a future input.

For $y(t) = x(2t)$ and $t < 0$, the output uses $2t < t$, which is the past. One output time that needs the future is enough to make a system non-causal.

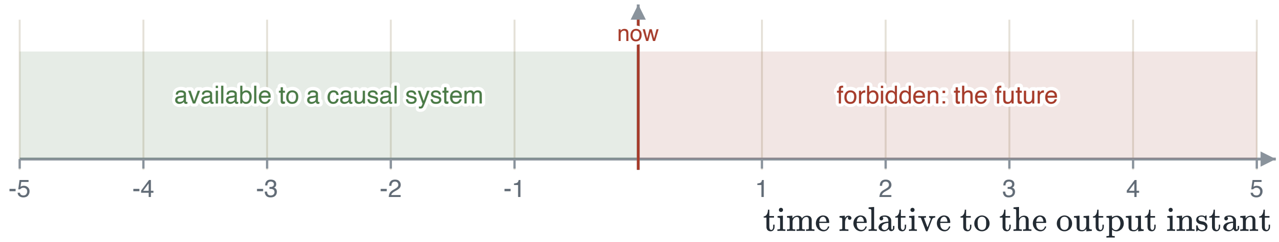

FIGURE 2.7 A causal system may use the past and the present of the input. It may not use the future.

A system that works while the signal arrives must be causal, because future input values are not yet available. A program that processes a stored recording may use later samples.

Stability

A signal is **bounded** if some finite constant B satisfies $|x(t)| \leq B$ for every t.

> ⓘ CRITERION
>
> A system is **BIBO stable** if every bounded input produces a bounded output. BIBO stands for bounded input, bounded output.

> ⚠ PROOF AND COUNTEREXAMPLE
>
> To prove stability, derive an output bound that holds for every bounded input. To disprove stability, find one bounded input that produces an unbounded output.

EXAMPLE 2.5

GIVEN $y(t) = 2x^2(t-1) + x(3t)$.

FIND Is the system BIBO stable?

METHOD Assume $|x(t)| \leq B$ for every t, and bound $|y|$ with the triangle inequality: the modulus of a sum is at most the sum of the moduli.

SOLUTION Start from the system rule, apply the triangle inequality, then use the input bound:

$$\begin{aligned} |y(t)| &= \left|2x^2(t-1) + x(3t)\right| \\ &\leq \left|2x^2(t-1)\right| + |x(3t)| \\ &= 2|x(t-1)|^2 + |x(3t)| \\ &\leq 2B^2 + B < \infty. \end{aligned}$$

The output has the finite bound $2B^2 + B$, so the system is stable.

CHECK A shift or a scale of the time axis does not change the set of values a signal takes. So $x(t-1)$ and $x(3t)$ are bounded by the same B.

EXAMPLE 2.6

GIVEN $y[n] = \sum_{k=-\infty}^{n} x[k]$, the accumulator.

FIND Is the system BIBO stable?

METHOD Look for one bounded input with an unbounded output.

SOLUTION Take $x[n] = u[n]$, so $|x[n]| \le 1$. The factor $u[k]$ removes every $k < 0$. For $n \ge 0$,

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{n} u[k] \\ &= \sum_{k=0}^{n} 1 \\ &= n + 1 \longrightarrow \infty. \end{aligned}$$

The input is bounded and the output is not. The system is **not** stable.

CHECK The input never exceeds 1, and the output passes every bound. That is exactly the failure that BIBO stability rules out.

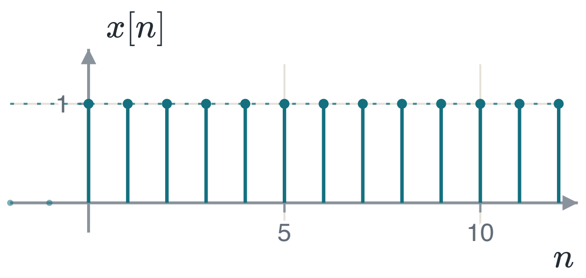

FIGURE 2.8 The bounded input $u[n]$.

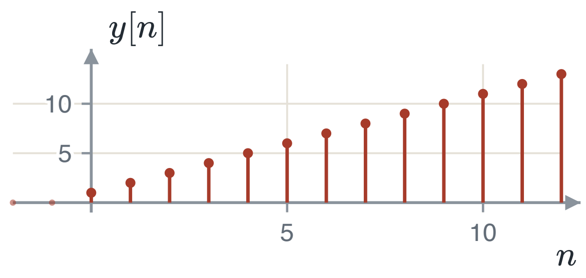

FIGURE 2.9 The unbounded output $n + 1$.

⚠ TWO SEPARATE TESTS

Causality and stability are separate tests. The accumulator is causal and not stable. The time reversal $y[n] = x[-n]$ is stable and not causal.

Time invariance

ⓘ CRITERION

If $x(t)$ produces $y(t)$, the system is **time invariant** when $x(t - t_0)$ produces $y(t - t_0)$, for every shift t_0.

The test compares two computed signals.

1. **Path 1.** Shift the input, then apply the system: $y_2 = S\{x(t - t_0)\}$.
2. **Path 2.** Apply the system, then shift the output: $y_1(t - t_0)$.

The system is time invariant exactly when the two agree, for every input and every t_0.

EXAMPLE 2.7

GIVEN $y(t) = \sin\bigl(x(t)\bigr)$.

FIND Is the system time invariant?

METHOD Compute both paths of the test for an arbitrary input x_1 and an arbitrary shift t_0.

SOLUTION Path 1 shifts the input and then applies the system:

$$\begin{aligned} x_2(t) &= x_1(t-t_0), \\ y_2(t) &= \sin\big(x_2(t)\big) = \sin\big(x_1(t-t_0)\big). \end{aligned}$$

Path 2 applies the system and then shifts its output:

$$\begin{aligned} y_1(t) &= \sin\big(x_1(t)\big), \\ y_1(t-t_0) &= \sin\big(x_1(t-t_0)\big). \end{aligned}$$

So $y_2(t) = y_1(t-t_0)$. The system is time invariant.

CHECK Both paths give the same expression, and neither step used a special input or a special t_0.

EXAMPLE 2.8

GIVEN $y[n] = n\,x[n]$.

FIND Is the system time invariant?

METHOD Look for a counterexample with an impulse input and a one-sample shift.

SOLUTION Take $x_1[n] = \delta[n]$. Path 2 applies the system and then shifts:

$$\begin{aligned} y_1[n] &= n\,\delta[n] = 0, \\ y_1[n-1] &= 0. \end{aligned}$$

Path 1 shifts the input first:

$$\begin{aligned} x_2[n] &= \delta[n-1], \\ y_2[n] &= n\,\delta[n-1] \\ &= 1\cdot\delta[n-1] = \delta[n-1]. \end{aligned}$$

The step $n\,\delta[n] = 0$ holds because $\delta[n]$ is non-zero only at $n = 0$, where the factor n is 0. The step $n\,\delta[n-1] = \delta[n-1]$ holds because $\delta[n-1]$ is non-zero only at $n = 1$. So $y_2[n] \neq y_1[n-1]$, and the system is **not** time invariant.

CHECK The two outputs differ at $n = 1$: path 1 gives 1 there, and path 2 gives 0. The explicit n in the rule does not move with the input.

EXAMPLE 2.9

GIVEN $y(t) = x(2t)$.

FIND Is the system time invariant?

METHOD Compute both paths for an arbitrary input. Then make the difference visible with the pulse $x_1(t) = 1$ for $|t| < 2$, zero elsewhere, and the shift $t_0 = 2$.

SOLUTION Path 1 shifts the input and then applies the system. The system replaces t by $2t$ in its input x_2:

$$x_2(t) = x_1(t - t_0),$$
$$y_2(t) = x_2(2t) = x_1(2t - t_0).$$

Path 2 applies the system and then shifts the output:

$$y_1(t) = x_1(2t),$$
$$y_1(t - t_0) = x_1\big(2(t - t_0)\big) = x_1(2t - 2t_0).$$

The two shifts differ: t_0 against $2t_0$. For the pulse with $t_0 = 2$, $y_2(t) = 1$ when $|2t - 2| < 2$. Add 2 to each side of $-2 < 2t - 2 < 2$ and divide by 2: $0 < t < 2$. Likewise $y_1(t - 2) = 1$ when $|2t - 4| < 2$, that is $-2 < 2t - 4 < 2$, so $1 < t < 3$. The two pulses differ, so the system is **not** time invariant.

CHECK The unshifted output $y_1(t) = x_1(2t)$ is the pulse $-1 < t < 1$. Path 1 moves it by $t_0/2 = 1$, to $0 < t < 2$. Path 2 moves it by $t_0 = 2$, to $1 < t < 3$.

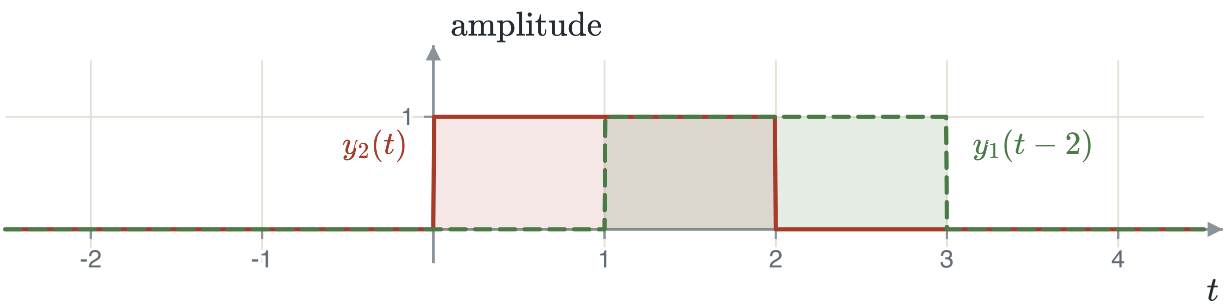

FIGURE 2.10 Example 2.9 with $t_0 = 2$. Path 1 gives the pulse on $0 < t < 2$; path 2 gives the dashed pulse on $1 < t < 3$.

> TWO PATTERNS THAT BREAK TIME INVARIANCE
>
> An explicit time variable in the rule, such as the n in $n\,x[n]$ or the $\cos(t)$ in $x(t)\cos(t)$, does not move with the input. A scaled or reversed argument, such as $x(2t)$ or $x[-n]$, changes the size or the direction of every shift. In either case, apply the two-path test.

Linearity

> CRITERION: SUPERPOSITION
>
> $$a\,x_1 + b\,x_2 \;\longrightarrow\; a\,y_1 + b\,y_2 \qquad \text{for all complex } a, b$$
>
> Here $x_1 \to y_1$ and $x_2 \to y_2$. The statement combines two conditions. **Additivity:** $x_1 + x_2 \to y_1 + y_2$. **Homogeneity:** $a\,x \to a\,y$ for every complex a.

EXAMPLE 2.10

GIVEN $y(t) = 2\pi\,x(t)$.

FIND Is the system linear?

METHOD Apply the system to a weighted sum and compare the result with the same weighted sum of the two outputs.

SOLUTION Let $x_3 = a\,x_1 + b\,x_2$. Apply the system and distribute the constant:

$$\begin{aligned} S\{x_3\} &= S\{a\,x_1 + b\,x_2\} \\ &= 2\pi(a\,x_1 + b\,x_2) \\ &= a(2\pi x_1) + b(2\pi x_2) \\ &= a\,y_1 + b\,y_2. \end{aligned}$$

Superposition holds, so the system is linear. It is also time invariant, so it is LTI (linear and time invariant).

CHECK The final expression is $a\,S\{x_1\} + b\,S\{x_2\}$ for arbitrary inputs and arbitrary complex scalars.

EXAMPLE 2.11

GIVEN $y[n] = \big(x[2n]\big)^2$.

FIND Is the system linear?

METHOD Compute both sides of the superposition test and compare them term by term.

SOLUTION Let $x_3[n] = a\,x_1[n] + b\,x_2[n]$. Combining first and then applying the system gives

$$\begin{aligned} y_3[n] &= \big(x_3[2n]\big)^2 \\ &= \big(a\,x_1[2n] + b\,x_2[2n]\big)^2 \\ &= a^2 x_1^2[2n] + 2ab\,x_1[2n]\,x_2[2n] + b^2 x_2^2[2n]. \end{aligned}$$

Applying the system first and then combining gives

$$a\,y_1[n] + b\,y_2[n] = a\,x_1^2[2n] + b\,x_2^2[2n].$$

The first expression has a cross term, and its powers of a and b differ. The system is **not** linear.

CHECK Scaling the input by a scales this output by a^2, so homogeneity fails on its own.

Complex scale factors

Additivity and homogeneity are separate conditions. A system can pass one and fail the other. Homogeneity must hold for every complex a, and a real a can hide a failure.

EXAMPLE 2.12

GIVEN $y[n] = \mathrm{Re}\{x[n]\}$, with complex inputs.

FIND Is the system linear?

METHOD Additivity holds, because the real part of a sum is the sum of the real parts. Test homogeneity with the complex scale factor $a = j$.

SOLUTION Write $x_1[n] = r[n] + j\,s[n]$, where $r[n]$ and $s[n]$ are its real and imaginary parts. Then $y_1[n] = r[n]$. Scale the input by j and use $j^2 = -1$:

$$\begin{aligned} x_2[n] &= j\,x_1[n] \\ &= j\,r[n] + j^2 s[n] \\ &= -s[n] + j\,r[n]. \end{aligned}$$

The output is its real part, $y_2[n] = -s[n]$. Homogeneity asks for $a\,y_1[n] = j\,r[n]$. A real $-s[n]$ cannot equal an imaginary $j\,r[n]$ unless both are zero. So $y_2 \neq a\,y_1$, and the system is **not** linear.

CHECK Take the single value $x_1 = 2 + j$. Then $y_1 = 2$ and $a\,y_1 = 2j$. The scaled input is $j(2 + j) = 2j + j^2 = -1 + 2j$, whose real part is $y_2 = -1 \neq 2j$. For a real a, $\mathrm{Re}\{a\,x\} = a\,\mathrm{Re}\{x\}$, so the test with a real scalar passes.

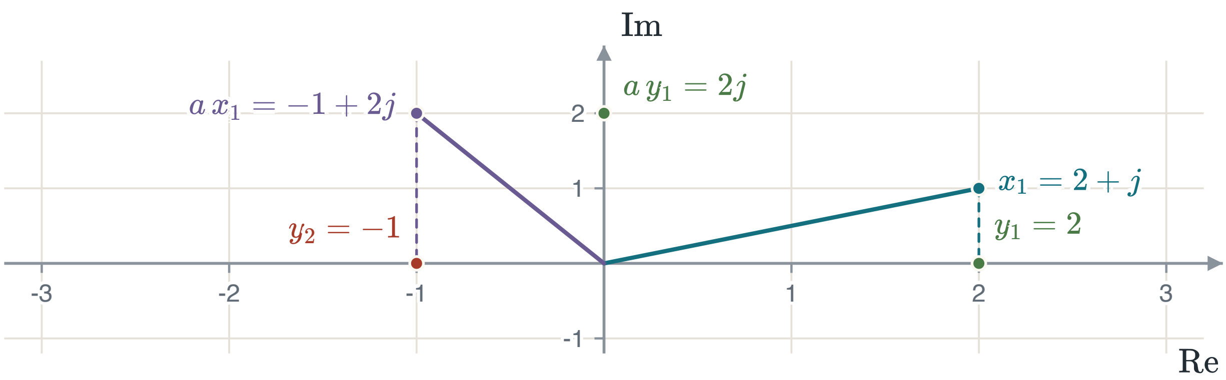

FIGURE 2.11 Example 2.12 in the complex plane with $a = j$. The output $y_2 = \mathrm{Re}\{a\,x_1\}$ is -1, but $a\,y_1$ is $2j$.

Incrementally linear systems

Every linear system maps the zero input to the zero output. Put $a = 0$ in homogeneity: $S\{0 \cdot x\} = 0 \cdot y = 0$. So a rule with an added constant cannot be linear. For example, $y(t) = x(t) + 1$ gives $y = 1$ for the zero input.

EXAMPLE 2.13

GIVEN $y[n] = 3\,x[n] + 2$, with the constant inputs $x_1[n] = 1$ and $x_2[n] = 2$.

FIND Does $x_1 + x_2$ give $y_1 + y_2$?

METHOD Compute the three outputs directly from the rule.

SOLUTION

$$\begin{aligned} y_1[n] &= 3 \cdot 1 + 2 = 5, \\ y_2[n] &= 3 \cdot 2 + 2 = 8, \\ y_3[n] &= 3 \cdot (1 + 2) + 2 = 11. \end{aligned}$$

But $y_1 + y_2 = 5 + 8 = 13 \neq 11$. Additivity fails, so the system is not linear.

CHECK The gap $13 - 11 = 2$ is the added constant. It enters y_3 once but $y_1 + y_2$ twice.

Split the output into a linear part and a part that does not depend on the input:

$$y[n] = \underbrace{3\,x[n]}_{\text{linear}} + \underbrace{2}_{y_0[n]}.$$

The term $y_0[n]$ is the **zero-input response**, the output when $x[n] = 0$. The difference of two outputs is linear in the difference of the inputs, because the constant cancels:

$$\begin{aligned} y_1[n] - y_2[n] &= \big(3\,x_1[n] + 2\big) - \big(3\,x_2[n] + 2\big) \\ &= 3\big(x_1[n] - x_2[n]\big). \end{aligned}$$

A system with this property is called **incrementally linear**. It is a linear system plus a zero-input response.

2.3 Classification in practice

Test an unfamiliar system in a fixed order. The order puts the tests that most often fail first.

1. **Time invariance.** Look for an explicit t or n, or a scaled or reversed argument. If you find one, apply the two-path test.
2. **Linearity.** Test with one scalar and one sum. Squares, products, $\sin(x)$, absolute values, saturation and added constants usually fail.
3. **Memory and causality.** Read every input argument. An argument different from t or n gives memory. An argument later than the output time gives non-causality.
4. **Stability, then invertibility.** Assume $|x| \le B$ and derive a finite output bound, or find a bounded input with an unbounded output. Then find an inverse formula, or two distinct inputs with the same output.

> ONE GENERAL IMPLICATION
>
> **Memoryless implies causal**, because a memoryless system uses only the present input. The converse fails: $y[n] = x[n-1]$ is causal and has memory. A causal system need not be stable, and a linear system need not be time invariant.

TABLE 2.3 Six properties of seven example systems.

SYSTEM	MEMORYLESS	INVERTIBLE	CAUSAL	STABLE	TIME INV.	LINEAR
$y(t) = 2\pi x(t)$	yes	yes	yes	yes	yes	yes
$y[n] = x[n-1]$	no	yes	yes	yes	yes	yes
$y(t) = x^2(t)$	yes	no	yes	yes	yes	no
$y[n] = n\,x[n]$	yes	no	yes	no	no	yes
$y[n] = \sum_{k\le n} x[k]$	no	yes	yes	no	yes	yes
$y[n] = x[-n]$	no	yes	no	yes	no	yes
$y[n] = (x[2n])^2$	no	no	no	yes	no	no

Read each property down its column. Every pattern of verdicts occurs, so each property needs its own test.

Everyday systems carry the same properties. An RC circuit, $v_C(t) = 5\,(1 - e^{-t/\tau})$ for a supply switched to 5 V, is causal and has memory. A guitar overdrive, $y(t) = \tanh\big(2x(t)\big)$, is memoryless and not linear. A savings account with 1 % a month, $y[n] = 1.01\,y[n-1] + x[n]$, is linear and time invariant, but a deposit of 100 each month makes its balance grow without bound. A digital echo, $y[n] = x[n] + 0.6\,x[n-8]$, is linear and time invariant.

2.4 Summary

TABLE 2.4 The six system properties: the definition of each and how to disprove it.

PROPERTY	DEFINITION	HOW TO DISPROVE IT
Memoryless	The output at t uses only the input at the same t.	Find an output that uses the input at another time.
Invertible	Distinct inputs give distinct outputs.	Find two distinct inputs with the same output.
Causal	The output at t uses only $x(\tau)$ for $\tau \le t$.	Find an output that uses a future input.
BIBO stable	Every bounded input gives a bounded output.	Find one bounded input with an unbounded output.
Time invariant	$x(t-t_0) \to y(t-t_0)$ for every t_0.	Find one input and one shift where the two paths differ.
Linear	$a\,x_1 + b\,x_2 \to a\,y_1 + b\,y_2$ for all complex a, b.	Find inputs and scalars where superposition fails.

WHAT CHAPTER 3 PROVES

If a system is linear and time invariant, its response to **one** input, the unit impulse, determines its response to **every** input.

Exercises

2.1 Classify $y(t) = x(t)\,u(t)$ against all six properties.
Answer: Memoryless, not invertible, causal, stable, not time invariant, linear.

2.2 Show that $y[n] = x[n] - x[n-1]$ is invertible on inputs that are zero for $n < 0$, and give the inverse.
Answer: $x[n] = \sum_{k=0}^{n} y[k]$.

2.3 Give a system that is linear and causal but not stable, and one that is stable and time invariant but not linear.

2.4 Is $y(t) = x(t/2)$ time invariant? Prove it or give a counterexample.
Answer: No. Path 1 gives $x(t/2 - t_0)$ and path 2 gives $x\big((t-t_0)/2\big)$.

CHAPTER

3

Linear time-invariant systems

This chapter develops a direct way to find the output of a linear time-invariant (LTI) system. Its response to one unit impulse describes the system completely. Convolution then uses that response to find the output for any input. The chapter ends with systems given by difference and differential equations.

3.1 Impulse response and the representation property

DEFINITION

The **impulse response** is the output when the input is a unit impulse: $x[n] = \delta[n]$ gives $y[n] = h[n]$. In continuous time, $x(t) = \delta(t)$ gives $y(t) = h(t)$.

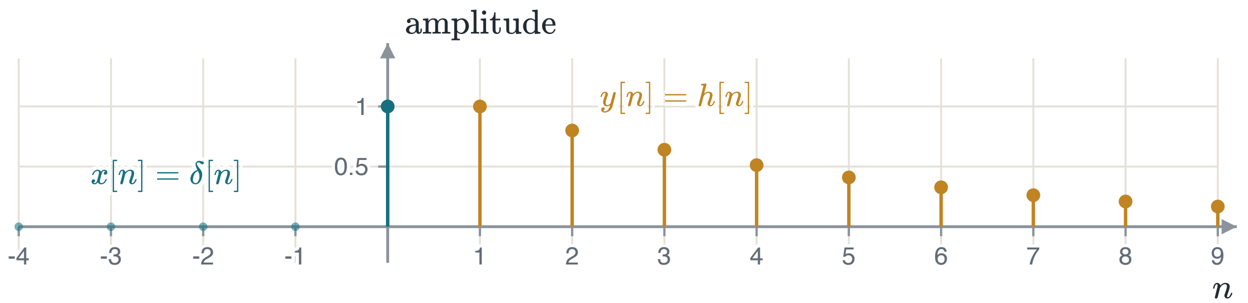

FIGURE 3.1 The input is one unit sample at $n = 0$. This system answers one step later, and its answer dies away.

For a general system, this experiment gives one input–output pair and nothing more. It does not determine the response to another input.

For an LTI system, one impulse response is enough. Time invariance gives the response to every shifted impulse: the input $\delta[n-3]$ gives the output $h[n-3]$. Linearity then gives the response to every weighted sum of shifted impulses. The next result shows that every discrete-time signal is such a sum. Therefore $h[n]$ determines the response to every input.

Every signal is a sum of impulses

Start from the sampling property of Chapter 1. A product with a shifted impulse keeps one sample, at $n = k$, with the weight $x[k]$:

$$x[n]\,\delta[n-k] = x[k]\,\delta[n-k].$$

Next, the sum of all shifted impulses is 1 at every n: in $\sum_k \delta[n-k]$ only the term $k = n$ is non-zero, and it equals 1. Multiply $x[n]$ by this sum, move $x[n]$ inside, and apply the sampling property to each term:

$$
\begin{aligned}
x[n] &= x[n] \underbrace{\sum_{k=-\infty}^{\infty} \delta[n-k]}_{=1} \\
&= \sum_{k=-\infty}^{\infty} x[n]\,\delta[n-k] \\
&= \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k].
\end{aligned}
$$

REPRESENTATION PROPERTY

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

Every sequence is a sum of shifted impulses. The impulse at $n = k$ carries the weight $x[k]$.

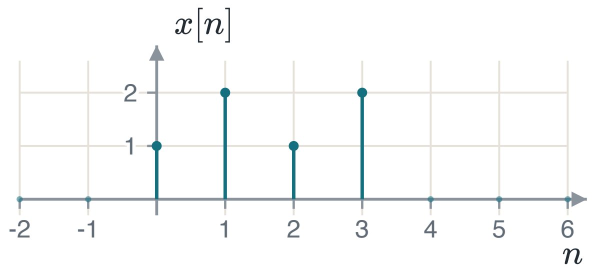

FIGURE 3.2 The sequence $x[n] = \{1, 2, 1, 2\}$ on $n = 0, \dots, 3$.

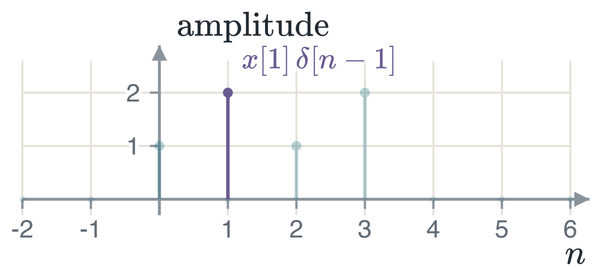

FIGURE 3.3 One term of the sum: $x[1]\,\delta[n-1] = 2\,\delta[n-1]$.

Read a signal off this form one index at a time. For $x[n] = 3\,\delta[n+1] - \delta[n-2]$, at $n = -1$ only the first impulse is non-zero:

$$
\begin{aligned}
x[-1] &= 3\,\delta[-1+1] - \delta[-1-2] \\
&= 3\,\delta[0] - \delta[-3] \\
&= 3 \cdot 1 - 0 = 3.
\end{aligned}
$$

3.2 The convolution sum

The purpose of this section is to obtain the output of an LTI system from $h[n]$. Start with the representation property, then apply the system to both sides.

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] \quad \longrightarrow \quad S \quad \longrightarrow \quad y[n] = ?$$

1. **Time invariance.** Since $\delta[n] \to h[n]$, also $\delta[n-k] \to h[n-k]$.
2. **Homogeneity.** The number $x[k]$ does not depend on n, so $x[k]\,\delta[n-k] \to x[k]\,h[n-k]$.
3. **Additivity.** The response to a sum is the sum of the responses.

Write the three moves as one chain. Apply the system to the representation. Take the sum outside the system by additivity, take each weight outside by homogeneity, and replace each shifted impulse response by time invariance:

$$
\begin{aligned}
y[n] &= S\Big\{ \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] \Big\} \\
&= \sum_{k=-\infty}^{\infty} S\{x[k]\,\delta[n-k]\} && \text{(additivity)} \\
&= \sum_{k=-\infty}^{\infty} x[k]\,S\{\delta[n-k]\} && \text{(homogeneity)} \\
&= \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] && \text{(time invariance).}
\end{aligned}
$$

CONVOLUTION SUM

$$
y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] \;=\; x[n] * h[n]
$$

The symbol $*$ denotes convolution.

⊗ CHECK THE SYSTEM BEFORE USING CONVOLUTION

Convolution gives the system output only when the system is linear and time invariant. The derivation uses time invariance to shift h, and linearity to scale and add the responses. For a system that is linear but not time invariant, the step $\delta[n-k] \to h[n-k]$ fails, and $x * h$ is not its output.

Either signal can be flipped

Replace the summation index by $m = n - k$, so $k = n - m$. As k runs over all integers, m also runs over all integers, only in the opposite direction. The sum has the same terms, so its value is unchanged:

$$
\begin{aligned}
\sum_{k=-\infty}^{\infty} x[k]\,h[n-k] &= \sum_{m=-\infty}^{\infty} x[n-m]\,h[n-(n-m)] \\
&= \sum_{m=-\infty}^{\infty} h[m]\,x[n-m] \;=\; h[n] * x[n].
\end{aligned}
$$

So convolution is commutative, and either factor may be the reversed and shifted one. Choose the signal whose support gives the simpler limits. For example, take $x[n] = \{1, 2, 1, 2\}$ on $n = 0, \dots, 3$ and $h[n] = \{1,\ 0.6,\ 0.3\}$ on $n = 0, 1, 2$, and evaluate both forms at $n = 3$. In the first form $x[0]h[3] = 0$, because $h[3] = 0$:

$$\begin{aligned}\sum_k x[k]\,h[3-k] &= x[1]h[2] + x[2]h[1] + x[3]h[0] \\ &= 2(0.3) + 1(0.6) + 2(1) = 3.2, \\ \sum_m h[m]\,x[3-m] &= h[0]x[3] + h[1]x[2] + h[2]x[1] \\ &= 1(2) + 0.6(1) + 0.3(2) = 3.2.\end{aligned}$$

The same pairs of samples meet in both sums. The second form also gives the effect of a shifted impulse at once. If $h[n] = \delta[n-2]$, only the term $m = 2$ survives:

$$x[n] * \delta[n-2] = \sum_{m=-\infty}^{\infty} \delta[m-2]\,x[n-m] = x[n-2].$$

Convolution with a shifted impulse shifts the signal.

Flip, shift, multiply, add

1. **Flip.** Reverse $h[k]$ to get $h[-k]$.
2. **Shift.** Move it by n to get $h[n-k]$.
3. **Multiply and add.** Form $x[k]\,h[n-k]$ and sum over k.
4. Repeat for every n.

In $h[n-k]$ the variable is k, and n is fixed. The sample $h[0]$ sits where $n - k = 0$, that is at $k = n$. With $h = \{1,\ 0.6,\ 0.3\}$ and $n = 3$, the sample $0.3 = h[2]$ sits where $3 - k = 2$, that is at $k = 1$.

> ⊗ DO NOT SKIP THE FLIP
>
> Without the flip the sum is $\sum_k x[k]\,h[k-n]$. This is the cross-correlation of x and h, not their convolution. A symmetric h hides this error, because reversal does not change it. Use an asymmetric example to check the construction.

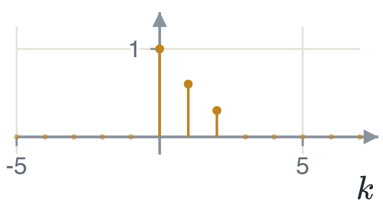

FIGURE 3.4 $h[k]$

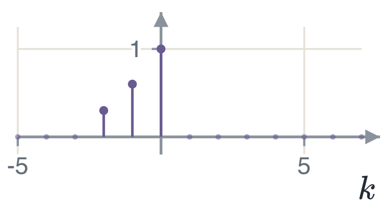

FIGURE 3.5 Flip: $h[-k]$

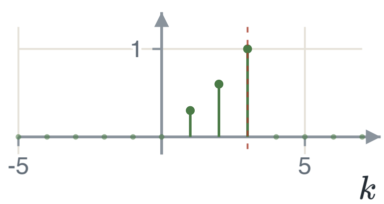

FIGURE 3.6 Shift: $h[3-k]$, with $h[0]$ at $k = 3$

✎ EXAMPLE 3.1

GIVEN $x[n] = \{1, 2, 1, 2\}$ on $n = 0, 1, 2, 3$, and $h[n] = \{1, 1\}$ on $n = 0, 1$. Both are zero elsewhere.

FIND $y[n] = x[n] * h[n]$.

METHOD Only four samples of x are non-zero, so the convolution sum has four terms. Each term is a copy of h, delayed to one sample of x and scaled by its value.

SOLUTION Start from the convolution sum and keep the terms $k = 0, 1, 2, 3$:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] \\ &= x[0]h[n] + x[1]h[n-1] + x[2]h[n-2] + x[3]h[n-3] \\ &= h[n] + 2h[n-1] + h[n-2] + 2h[n-3]. \end{aligned}$$

Now evaluate at each n. Since $h[m] = 1$ only for $m = 0$ and $m = 1$, the factor $h[n-k]$ is non-zero only when $n - k = 0$ or $n - k = 1$. Keep only those terms:

$$\begin{aligned} y[0] &= x[0]h[0] = 1 \cdot 1 = 1, \\ y[1] &= x[0]h[1] + x[1]h[0] = 1 \cdot 1 + 2 \cdot 1 = 3, \\ y[2] &= x[1]h[1] + x[2]h[0] = 2 \cdot 1 + 1 \cdot 1 = 3, \\ y[3] &= x[2]h[1] + x[3]h[0] = 1 \cdot 1 + 2 \cdot 1 = 3, \\ y[4] &= x[3]h[1] = 2 \cdot 1 = 2. \end{aligned}$$

For $n < 0$ and $n > 4$ no term survives, so $y[n] = 0$. The result is $y = \{1, 3, 3, 3, 2\}$ on $n = 0, \dots, 4$.

CHECK The support must hold $4 + 2 - 1 = 5$ samples, from 0 to 4, which it does. The sums must multiply: $\sum_n y[n] = 1 + 3 + 3 + 3 + 2 = 12$, and $\left(\sum_n x[n]\right)\left(\sum_n h[n]\right) = 6 \times 2 = 12$. Also, $h = \{1, 1\}$ is a two-point moving sum, so each output adds two neighbouring inputs: 1, $1 + 2$, $2 + 1$, $1 + 2$, 2.

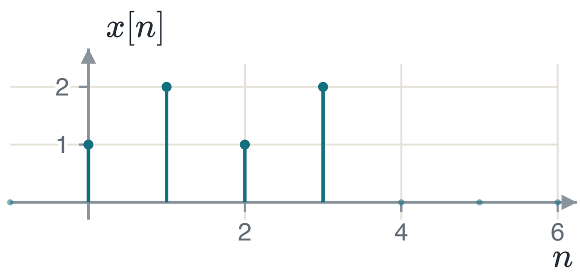

FIGURE 3.7 $x[n]$

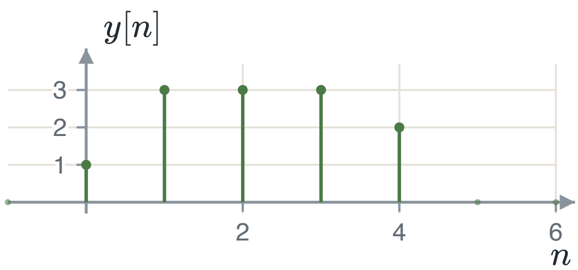

FIGURE 3.8 $y[n] = x[n] * h[n]$

EXAMPLE 3.2

GIVEN $x[n] = \left(\frac{1}{2}\right)^n u[n]$ and $h[n] = u[n]$.

FIND $y[n] = x[n] * h[n]$.

METHOD Use the supports to set the summation limits. $u[k]$ keeps only $k \geq 0$, and $u[n-k]$ keeps only $k \leq n$. The overlap changes when n changes sign, so treat $n < 0$ and $n \geq 0$ separately.

SOLUTION Substitute the two signals into the convolution sum:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = \sum_{k=-\infty}^{\infty} \left(\tfrac{1}{2}\right)^k u[k]\, u[n-k].$$

The factor $u[k]$ is 1 only for $k \geq 0$. The factor $u[n-k]$ is 1 only for $n-k \geq 0$, that is for $k \leq n$. A term survives only when both hold.

Case 1, $n < 0$. No k satisfies both $k \geq 0$ and $k \leq n < 0$. No term survives, so $y[n] = 0$.

Case 2, $n \geq 0$. The surviving terms are $0 \leq k \leq n$, and on this range both steps equal 1:

$$y[n] = \sum_{k=0}^{n} \left(\tfrac{1}{2}\right)^k.$$

This is a finite geometric sum with first term 1, ratio $r = \frac{1}{2}$ and $n+1$ terms. Apply the formula in the box below with $a = 1$ and $m = 0$:

$$\begin{aligned} y[n] &= \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}} \\ &= \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{\frac{1}{2}} \\ &= 2 - 2\left(\tfrac{1}{2}\right)^{n+1} \\ &= 2 - \left(\tfrac{1}{2}\right)^{n}. \end{aligned}$$

The last line uses $2\left(\frac{1}{2}\right)^{n+1} = \left(\frac{1}{2}\right)^n$: the factor 2 cancels one factor of $\frac{1}{2}$.
Together, $y[n] = \left(2 - \left(\frac{1}{2}\right)^n\right) u[n]$.

CHECK

At $n = 0$ the formula gives $2 - 1 = 1$, and direct substitution gives $x[0]h[0] = 1 \cdot 1 = 1$. As $n \to \infty$, $y[n] \to 2$, the total sum of the input: $\sum_{k \geq 0} \left(\frac{1}{2}\right)^k = 1/(1 - \frac{1}{2}) = 2$. This fits $h[n] = u[n]$, which forms the running sum of its input.

ⓘ GEOMETRIC SUM

$\displaystyle\sum_{k=m}^{n} a\, r^k = \frac{a\left(r^m - r^{n+1}\right)}{1-r}$. The finite sum needs only $r \neq 1$. The condition $|r| < 1$ is needed only when the sum runs to infinity.

⚠ THE ACCUMULATOR

In Example 3.2 the input has a finite sum, so the output stays bounded. The accumulator $h[n] = u[n]$ is still not BIBO stable: Example 2.6 gave it the bounded input $u[n]$ and an unbounded output.

3.3 The convolution integral

The continuous-time derivation has the same purpose and uses the same three LTI moves. First, the sifting property represents x as a continuum of weighted impulses:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,\mathrm{d}\tau.$$

This form comes from the sifting property of Chapter 1, $x(t_0) = \int x(t)\,\delta(t-t_0)\,\mathrm{d}t$. Rename the integration variable t as τ, and the instant t_0 as t. This gives $x(t) = \int x(\tau)\,\delta(\tau - t)\,\mathrm{d}\tau$. The impulse is even, $\delta(\tau - t) = \delta(t-\tau)$, which gives the form above.

Pulses of width Δ

A staircase shows where the continuum of impulses comes from. Let $\delta_\Delta(t)$ be a pulse of height $1/\Delta$ on $0 \le t < \Delta$, so its area is 1. On the interval $k\Delta \le t < (k+1)\Delta$, the staircase below keeps only its term k, which equals $x(k\Delta)\cdot\frac{1}{\Delta}\cdot\Delta = x(k\Delta)$:

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\,\delta_\Delta(t-k\Delta)\,\Delta.$$

Each pulse has area $x(k\Delta)\,\Delta$. Let $\hat{h}_\Delta(t)$ be the response of the system to $\delta_\Delta(t)$. Time invariance and linearity give the response to the staircase. As $\Delta \to 0$, $k\Delta$ becomes the continuous variable τ, Δ becomes $\mathrm{d}\tau$, the pulse becomes the impulse, $\hat{h}_\Delta$ becomes h, and the sum becomes an integral:

$$\hat{y}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\,\hat{h}_\Delta(t-k\Delta)\,\Delta \xrightarrow{\Delta\to 0} \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,\mathrm{d}\tau.$$

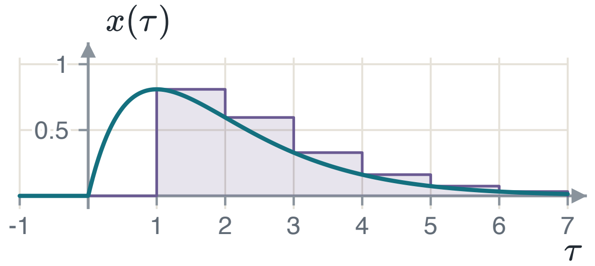

FIGURE 3.9 Pulses of width $\Delta = 1$.

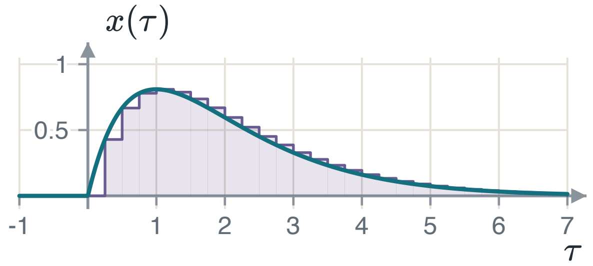

FIGURE 3.10 Pulses of width $\Delta = 0.25$: the staircase follows $x(\tau)$ closely.

The direct derivation applies the system to the sifting form. Additivity takes the integral outside the system, because an integral is a limit of sums. Homogeneity takes the weight $x(\tau)\,\mathrm{d}\tau$ outside, because it does not depend on t. Time invariance replaces $S\{\delta(t-\tau)\}$ by $h(t-\tau)$:

$$\begin{aligned} y(t) &= S\Big\{\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,\mathrm{d}\tau\Big\} \\ &= \int_{-\infty}^{\infty} x(\tau)\,S\big\{\delta(t-\tau)\big\}\,\mathrm{d}\tau \qquad \text{(additivity, homogeneity)} \\ &= \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,\mathrm{d}\tau \qquad \text{(time invariance).} \end{aligned}$$

CONVOLUTION INTEGRAL

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, \mathrm{d}\tau \;=\; x(t) * h(t)$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, \mathrm{d}\tau$$

The two forms are equal. Choose the form that makes the support conditions easier to write.

To pass from the first form to the second, substitute $\sigma = t - \tau$. Then $\tau = t - \sigma$ and $\mathrm{d}\tau = -\mathrm{d}\sigma$. As τ goes from $-\infty$ to $+\infty$, σ goes from $+\infty$ to $-\infty$. Swapping the limits back removes the minus sign:

$$\begin{aligned}\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, \mathrm{d}\tau &= \int_{+\infty}^{-\infty} x(t-\sigma)\, h(\sigma)\, (-\mathrm{d}\sigma) \\ &= \int_{-\infty}^{\infty} h(\sigma)\, x(t-\sigma)\, \mathrm{d}\sigma.\end{aligned}$$

The name of the integration variable does not matter, so σ can be written as τ again. This is the second form, and it shows that $x * h = h * x$.

A shifted impulse again gives a shift. If $h(t) = \delta(t-2)$, the impulse $\delta(t-\tau-2)$ sits where $t - \tau - 2 = 0$, that is at $\tau = t - 2$, and sifting keeps the value of x there:

$$x(t) * \delta(t-2) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau-2)\, \mathrm{d}\tau = x(t-2).$$

⚠ SHIFT, THEN REVERSE

Build a reversed and shifted signal in a fixed order: apply the shift before the reversal. For $\delta(-t+5)$, first form $v(t) = \delta(t+5)$, an impulse at $t = -5$. Then form $v(-t) = \delta(-t+5)$, which moves it to $t = +5$. An impulse sits where its argument is zero, so $\delta(3-t)$ sits at $t = 3$. Writing both moves prevents a sign error in the integration limits.

EXAMPLE 3.3

GIVEN $x(t) = e^{2t}u(-t)$ and $h(t) = u(t-3)$.

FIND $y(t) = x(t) * h(t)$.

METHOD Use the two support conditions to set the integration limit. $x(\tau)$ is non-zero for $\tau \le 0$, and $h(t-\tau) = u(t-\tau-3)$ is non-zero for $\tau \le t-3$. The overlap ends at $\min(0,\, t-3)$, so the upper limit changes at $t = 3$.

SOLUTION Substitute the two signals into the convolution integral:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, \mathrm{d}\tau = \int_{-\infty}^{\infty} e^{2\tau} u(-\tau)\, u(t-\tau-3)\, \mathrm{d}\tau.$$

The factor $u(-\tau)$ is 1 only for $\tau \le 0$. The factor $u(t-\tau-3)$ is 1 only for $t - \tau - 3 \ge 0$, that is for $\tau \le t - 3$. Both must hold, so the integrand is $e^{2\tau}$ for $\tau \le \min(0,\, t-3)$ and zero elsewhere. The antiderivative of $e^{2\tau}$ is $\frac{1}{2}e^{2\tau}$, and $e^{2\tau} \to 0$ as $\tau \to -\infty$, so the lower limit contributes zero.

Case 1, $t < 3$. Here $t - 3 < 0$, so the upper limit is $t - 3$:

$$\begin{aligned} y(t) &= \int_{-\infty}^{t-3} e^{2\tau}\,\mathrm{d}\tau \\ &= \left[\tfrac{1}{2}e^{2\tau}\right]_{-\infty}^{t-3} \\ &= \tfrac{1}{2}e^{2(t-3)} - 0 = \tfrac{1}{2}e^{2(t-3)}. \end{aligned}$$

Case 2, $t > 3$. Here $t - 3 > 0$, so the upper limit is 0 and the whole of x is covered:

$$\begin{aligned} y(t) &= \int_{-\infty}^{0} e^{2\tau}\,\mathrm{d}\tau \\ &= \left[\tfrac{1}{2}e^{2\tau}\right]_{-\infty}^{0} \\ &= \tfrac{1}{2}e^{0} - 0 = \tfrac{1}{2}. \end{aligned}$$

Together, $y(t) = \frac{1}{2}e^{2(t-3)}$ for $t < 3$ and $y(t) = \frac{1}{2}$ for $t > 3$.

CHECK At $t = 3$ the first branch gives $\frac{1}{2}e^{0} = \frac{1}{2}$, the value of the second, so the branches join continuously. The final value is the area of x, $\int_{-\infty}^{0} e^{2\tau}\,\mathrm{d}\tau = \frac{1}{2}$. This fits the system: the delayed step adds up the area of x until $t - 3$, and after $t = 3$ it holds all of it.

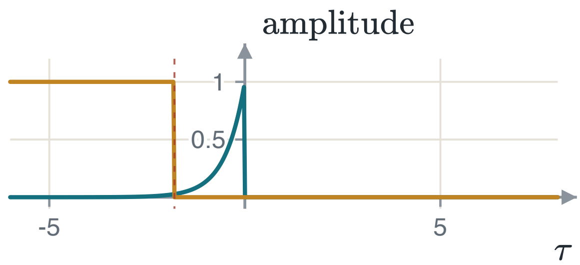

FIGURE 3.11 Case 1, $t < 3$: $x(\tau)$ and $h(t - \tau)$. The shaded area is $y(t)$.

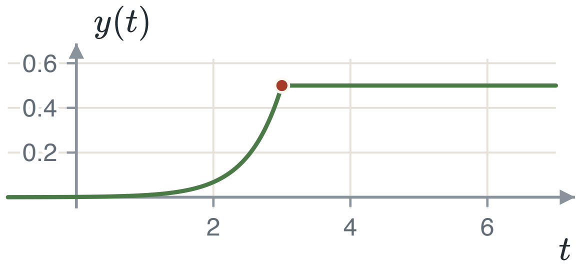

FIGURE 3.12 The result $y(t)$, continuous at $t = 3$.

EXAMPLE 3.4

GIVEN $x(t) = 1$ on $0 < t < 1$ and zero elsewhere. $h(t) = t$ on $0 < t < 2$ and zero elsewhere.

FIND $y(t) = x(t) * h(t)$.

METHOD Use $y(t) = \int h(\tau)\,x(t - \tau)\,\mathrm{d}\tau$. This form reverses the rectangle, whose constant height makes the overlap easy to read. Its support becomes the window $t - 1 < \tau < t$, of width 1, which moves across the ramp on $0 < \tau < 2$.

SOLUTION Substitute into the chosen form:

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\,x(t - \tau)\,\mathrm{d}\tau.$$

Here $h(\tau) = \tau$ for $0 < \tau < 2$. The factor $x(t-\tau)$ equals 1 when $0 < t - \tau < 1$. Solve this pair of inequalities for τ: subtract t to get $-t < -\tau < 1 - t$, then multiply by -1, which reverses both inequalities: $t - 1 < \tau < t$. So the integrand is τ on the overlap of $0 < \tau < 2$ with $t - 1 < \tau < t$, and zero elsewhere.

The window has the moving edges $t-1$ and t. The ramp has the fixed edges 0 and 2. Set each moving edge equal to each fixed edge: $t - 1 = 0$, $t - 1 = 2$, $t = 0$ and $t = 2$. This gives $t = 0, 1, 2, 3$, which divide the calculation into five cases. In every case the antiderivative of τ is $\frac{1}{2}\tau^2$.

$t < 0$: the window lies left of the ramp. There is no overlap, so $y = 0$.

$0 < t < 1$: the left edge $t - 1$ is below 0, so the overlap is $0 < \tau < t$:

$$y(t) = \int_0^t \tau \, \mathrm{d}\tau = \left[\tfrac{1}{2}\tau^2\right]_0^t = \tfrac{1}{2}t^2 - 0 = \tfrac{1}{2}t^2.$$

$1 < t < 2$: the whole window lies on the ramp, so the overlap is $t - 1 < \tau < t$:

$$\begin{aligned} y(t) &= \int_{t-1}^t \tau \, \mathrm{d}\tau = \left[\tfrac{1}{2}\tau^2\right]_{t-1}^t \\ &= \tfrac{1}{2}t^2 - \tfrac{1}{2}(t-1)^2 \\ &= \tfrac{1}{2}t^2 - \tfrac{1}{2}\left(t^2 - 2t + 1\right) \\ &= t - \tfrac{1}{2}. \end{aligned}$$

$2 < t < 3$: the right edge t is past 2, so the overlap is $t - 1 < \tau < 2$:

$$\begin{aligned} y(t) &= \int_{t-1}^2 \tau \, \mathrm{d}\tau = \left[\tfrac{1}{2}\tau^2\right]_{t-1}^2 \\ &= \tfrac{1}{2} \cdot 4 - \tfrac{1}{2}(t-1)^2 \\ &= 2 - \tfrac{1}{2}\left(t^2 - 2t + 1\right) \\ &= -\tfrac{1}{2}t^2 + t + \tfrac{3}{2}. \end{aligned}$$

$t > 3$: the window lies right of the ramp. There is no overlap, so $y = 0$.

CHECK **Continuity.** At $t = 1$: $\frac{1}{2} \cdot 1^2 = \frac{1}{2}$ and $1 - \frac{1}{2} = \frac{1}{2}$. At $t = 2$: $2 - \frac{1}{2} = \frac{3}{2}$ and $-\frac{1}{2} \cdot 4 + 2 + \frac{3}{2} = \frac{3}{2}$. At $t = 3$: $-\frac{1}{2} \cdot 9 + 3 + \frac{3}{2} = 0$. Every pair of branches agrees. **Support.** The supports $[0, 1]$ and $[0, 2]$ add to $[0, 3]$. **Area.** The areas must multiply: $\int y = \left(\int x\right)\left(\int h\right) = 1 \times 2 = 2$. Integrate each branch:

$$\begin{aligned} \int_0^1 \tfrac{1}{2}t^2 \, \mathrm{d}t &= \left[\tfrac{1}{6}t^3\right]_0^1 = \tfrac{1}{6}, \\ \int_1^2 \left(t - \tfrac{1}{2}\right) \mathrm{d}t &= \left[\tfrac{1}{2}t^2 - \tfrac{1}{2}t\right]_1^2 = (2 - 1) - \left(\tfrac{1}{2} - \tfrac{1}{2}\right) = 1, \\ \int_2^3 \left(-\tfrac{1}{2}t^2 + t + \tfrac{3}{2}\right) \mathrm{d}t &= \left[-\tfrac{1}{6}t^3 + \tfrac{1}{2}t^2 + \tfrac{3}{2}t\right]_2^3 \\ &= \left(-\tfrac{27}{6} + \tfrac{9}{2} + \tfrac{9}{2}\right) - \left(-\tfrac{8}{6} + 2 + 3\right) = \tfrac{9}{2} - \tfrac{11}{3} = \tfrac{5}{6}. \end{aligned}$$

The total is $\frac{1}{6} + 1 + \frac{5}{6} = 2$, as required.

FIND THE BOUNDARIES BEFORE INTEGRATING

List the moving edges and the fixed edges, then set each moving edge equal to each fixed edge. The resulting values divide the calculation into all the cases it needs.

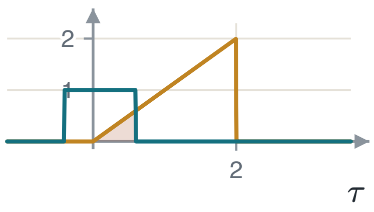

FIGURE 3.13 $0 < t < 1$

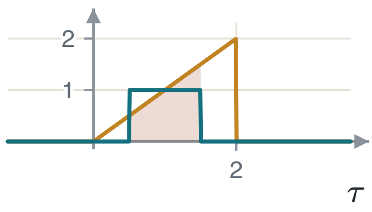

FIGURE 3.14 $1 < t < 2$

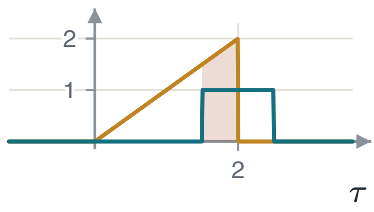

FIGURE 3.15 $2 < t < 3$

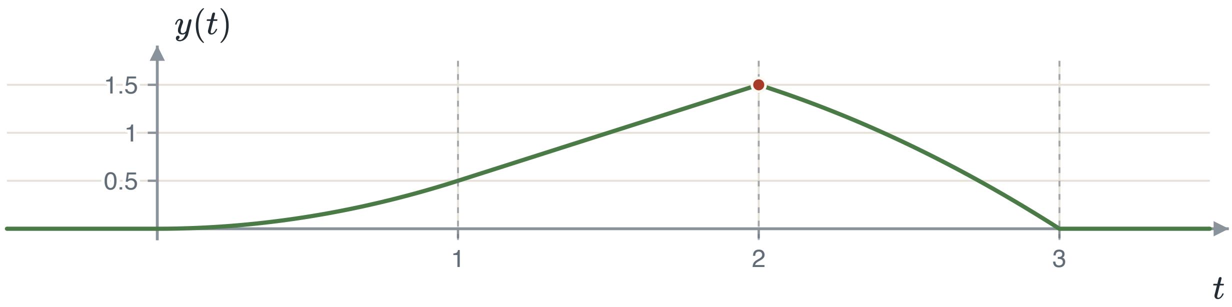

FIGURE 3.16 The complete result. The peak is 1.5 at $t = 2$. A unit-width rectangle adds up one second of the input, so the output rises while the window fills and falls as it leaves the ramp.

A convolution checklist

1. Confirm that the system is linear and time invariant. Convolution gives the system output only under this condition.
2. Choose the factor whose reversal gives the simpler support conditions.
3. Write the support of each factor as an inequality in the dummy variable.
4. Set each moving edge equal to each fixed edge to list every case boundary, before integrating anything.
5. Integrate or sum, case by case.
6. Check continuity at every boundary, check that the supports add, and check that the total areas or sums multiply.

WHAT EACH CHECK FINDS

A mismatch at a case boundary points to a wrong limit. A wrong output support points to an error in a shift, a reversal or a support condition. A wrong total area or sum points to an error in the integrand or the summand.

3.4 Properties of convolution

TABLE 3.1 Properties of convolution and what they mean for interconnections.

PROPERTY	STATEMENT	WHAT IT MEANS FOR INTERCONNECTIONS
Commutative	$x * h = h * x$	The input and the impulse response play the same role in the algebra.
Distributive	$x * (h_1 + h_2) = x * h_1 + x * h_2$	Two systems in **parallel**, outputs added, act as one system with impulse response $h_1 + h_2$.
Associative	$x * (h_1 * h_2) = (x * h_1) * h_2$	Two systems in **cascade** act as one system with impulse response $h_1 * h_2$.

Commutativity was derived in Sections 3.2 and 3.3 by a change of variable. The distributive property follows because a sum of two sequences can be split term by term:

$$\begin{aligned}\big(x * (h_1 + h_2)\big)[n] &= \sum_{k=-\infty}^{\infty} x[k]\big(h_1[n-k] + h_2[n-k]\big)\\ &= \sum_{k=-\infty}^{\infty} x[k]\, h_1[n-k] + \sum_{k=-\infty}^{\infty} x[k]\, h_2[n-k]\\ &= (x * h_1)[n] + (x * h_2)[n].\end{aligned}$$

So parallel paths add their impulse responses. For example, $h_1[n] = \delta[n]$ and $h_2[n] = \delta[n-1]$ in parallel act as one system with $h[n] = \delta[n] + \delta[n-1]$. The same steps with integrals in place of sums give the continuous-time result.

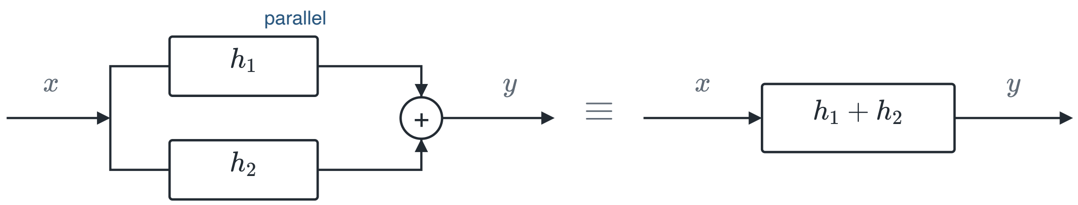

FIGURE 3.17 Two systems in parallel, with their outputs added, act as one system with impulse response $h_1 + h_2$.

For the associative property, write the cascade $(x * h_1) * h_2$ as a double sum, exchange the order of the two sums, and change the inner index. Let $w = x * h_1$ be the output of the first system. The inner sum over m uses the substitution $p = m - k$, so $m = p + k$ and $n - m = (n - k) - p$:

$$\begin{aligned}\big((x * h_1) * h_2\big)[n] &= \sum_{m=-\infty}^{\infty} w[m]\, h_2[n-m] \\ &= \sum_{m=-\infty}^{\infty} \Big(\sum_{k=-\infty}^{\infty} x[k]\, h_1[m-k]\Big) h_2[n-m] \\ &= \sum_{k=-\infty}^{\infty} x[k] \sum_{m=-\infty}^{\infty} h_1[m-k]\, h_2[n-m] \\ &= \sum_{k=-\infty}^{\infty} x[k] \sum_{p=-\infty}^{\infty} h_1[p]\, h_2[(n-k)-p] \\ &= \sum_{k=-\infty}^{\infty} x[k]\, (h_1 * h_2)[n-k] \\ &= \big(x * (h_1 * h_2)\big)[n].\end{aligned}$$

The third line moves $x[k]$ outside the sum over m, because $x[k]$ does not depend on m. The fifth line recognises the inner sum as the convolution $h_1 * h_2$ evaluated at $n-k$. Exchanging the two sums is allowed when the double sum converges absolutely, which holds for finite-length or absolutely summable sequences.

With commutativity, associativity shows that the order of two LTI systems in cascade does not change the combined impulse response.

> ⊗ LTI SYSTEMS ONLY
>
> Reordering a cascade needs both systems to be linear and time invariant. A saturating amplifier followed by a filter is in general not the same system as the filter followed by the amplifier.

Inverse systems

An LTI system with impulse response g is the inverse of the system with impulse response h when the cascade returns its input unchanged. The identity system has impulse response δ, so the condition is

$$h * g = \delta.$$

EXAMPLE 3.5

GIVEN $h_1[n] = \delta[n] - \delta[n-1]$ and $h_2[n] = u[n]$ are connected in cascade.

FIND The impulse response $h_1 * h_2$ of the cascade.

METHOD Distribute the convolution over the two impulses of h_1. Convolution with $\delta[n-n_0]$ shifts a signal by n_0 (Section 3.2).

SOLUTION

$$\begin{aligned}h_1 * h_2 &= \big(\delta[n] - \delta[n-1]\big) * u[n] \\ &= \delta[n] * u[n] - \delta[n-1] * u[n] \\ &= u[n] - u[n-1].\end{aligned}$$

Evaluate the difference region by region. For $n < 0$ both steps are 0. At $n = 0$ it is $1 - 0 = 1$. For $n \geq 1$ it is $1 - 1 = 0$. So $h_1 * h_2 = \delta[n]$: the two systems are inverses of each other.

CHECK h_2 is the accumulator and h_1 is the first difference, and Section 2.2 showed that the first difference undoes the accumulator.

System properties from the impulse response

For an LTI system, each property test of Chapter 2 becomes a test on h.

TABLE 3.2 System properties read from the impulse response of an LTI system.

PROPERTY	CRITERION	REASON
Memoryless	$h[n] = a\,\delta[n]$, or $h(t) = a\,\delta(t)$	Then $y[n] = a\,x[n]$, a pure gain.
Invertible	$h * g = \delta$ for some g	g is the impulse response of the inverse system.
Causal	$h[n] = 0$ for $n < 0$, or $h(t) = 0$ for $t < 0$	The output uses only present and past inputs.
BIBO stable	$\sum_k \lvert h[k]\rvert < \infty$, or $\int \lvert h(t)\rvert \,\mathrm{d}t < \infty$	See the proof below.

Memoryless. Substitute $h[n-k] = a\,\delta[n-k]$ into the convolution sum. The impulse keeps only the term $k = n$:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[k]\,a\,\delta[n-k] \\ &= a\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] \\ &= a\,x[n]. \end{aligned}$$

A non-zero value of h at some $n \neq 0$ would make the output read the input at another time.

Causal. Use the second form of the sum, $y[n] = \sum_k h[k]\,x[n-k]$. If $h[k] = 0$ for $k < 0$, every term with a negative k vanishes and the sum starts at $k = 0$:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k]\,x[n-k] \\ &= \sum_{k=0}^{\infty} h[k]\,x[n-k] \\ &= h[0]x[n] + h[1]x[n-1] + h[2]x[n-2] + \cdots \end{aligned}$$

Every input index $n-k$ with $k \geq 0$ satisfies $n - k \leq n$, so the output uses only present and past inputs. For example, $h[n] = \delta[n+1]$ has $h[-1] = 1 \neq 0$, and its output is $y[n] = x[n+1]$, a future sample. That system is not causal.

Stable, sufficiency. Assume $|x[n]| \leq B$ for all n. Take the modulus of the convolution sum, apply the triangle inequality, and then use the input bound on every term:

$$
\begin{aligned}
|y[n]| &= \left|\sum_{k=-\infty}^{\infty} h[k]\, x[n-k]\right| \\
&\le \sum_{k=-\infty}^{\infty} |h[k]|\,|x[n-k]| && \text{(triangle inequality)} \\
&\le \sum_{k=-\infty}^{\infty} |h[k]|\, B && (\text{since } |x[n-k]| \le B) \\
&= B \sum_{k=-\infty}^{\infty} |h[k]| < \infty.
\end{aligned}
$$

An absolutely summable $h[n]$ therefore maps every bounded input to a bounded output. The bound $B\sum_k |h[k]|$ does not depend on n. For example, $h[n] = \left(-\frac{1}{2}\right)^n u[n]$ gives the geometric sum $\sum_{n\ge 0} \left(\frac{1}{2}\right)^n = 1/(1-\frac{1}{2}) = 2 < \infty$, so that system is stable.

Stable, necessity. Suppose instead that $\sum_k |h[k]|$ is infinite. Choose the input $x[n] = \operatorname{sgn} h[-n]$, where sgn is $+1$ for a positive argument, -1 for a negative one and 0 at zero. This input is bounded, because $|x[n]| \le 1$. Evaluate the output at $n = 0$. Since $x[-k] = \operatorname{sgn} h[k]$, and a number times its own sign is its modulus:

$$
\begin{aligned}
y[0] &= \sum_{k=-\infty}^{\infty} h[k]\, x[0-k] \\
&= \sum_{k=-\infty}^{\infty} h[k] \operatorname{sgn} h[k] \\
&= \sum_{k=-\infty}^{\infty} |h[k]| \to \infty.
\end{aligned}
$$

A bounded input gives an unbounded output, so the system is not stable. Absolute summability is therefore also necessary. The same input reaches the sum when it is finite. For $h[n] = (-0.8)^n u[n]$, the input $x[n] = \operatorname{sgn} h[-n]$ is $(-1)^n$ for $n \le 0$ and 0 for $n > 0$, and

$$
y[0] = \sum_{k=0}^{\infty} (-0.8)^k (-1)^k = \sum_{k=0}^{\infty} 0.8^k = \frac{1}{1-0.8} = 5.
$$

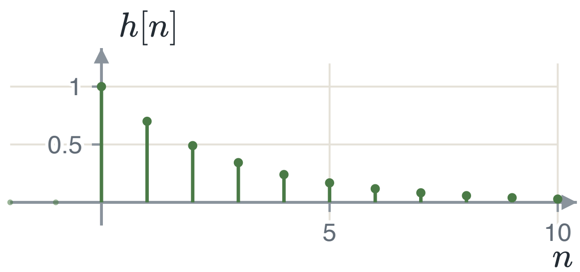

FIGURE 3.18 $h[n] = 0.7^n u[n]$. Here $\sum_n |h[n]| = \sum_{n=0}^{\infty} 0.7^n = \frac{1}{1-0.7} = \frac{10}{3} < \infty$: stable and causal.

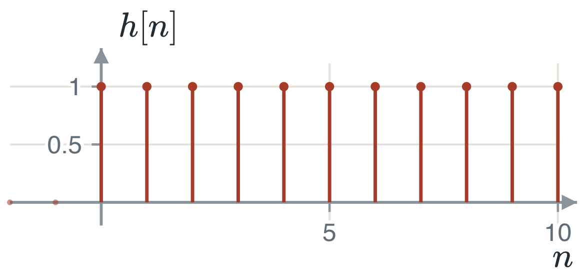

FIGURE 3.19 $h[n] = u[n]$, the accumulator. Here $\sum_n |h[n]|$ diverges: causal but not stable.

The step response

The **step response** s is the output of an LTI system when the input is the unit step. By the convolution integral, $s = h * u$. The factor $u(t-\tau)$ is 1 for $\tau < t$ and 0 for $\tau > t$, so it cuts the integral off at $\tau = t$:

$$\begin{aligned} s(t) &= \int_{-\infty}^{\infty} h(\tau)\, u(t-\tau)\, \mathrm{d}\tau \\ &= \int_{-\infty}^{t} h(\tau)\, \mathrm{d}\tau. \end{aligned}$$

The step response is the running integral of h: its value at t is the area of h up to t. In discrete time, $u[n-k]$ is 1 for $k \le n$, and the step response is the running sum of h:

$$\begin{aligned} s[n] &= \sum_{k=-\infty}^{\infty} h[k]\, u[n-k] \\ &= \sum_{k=-\infty}^{n} h[k]. \end{aligned}$$

Both operations can be undone. The derivative of a running integral is its integrand, and a first difference removes all but the last term of a running sum:

$$\begin{aligned} \frac{\mathrm{d}s(t)}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^{t} h(\tau)\, \mathrm{d}\tau = h(t), \\ s[n] - s[n-1] &= \sum_{k=-\infty}^{n} h[k] - \sum_{k=-\infty}^{n-1} h[k] = h[n]. \end{aligned}$$

A step is easier to apply than an impulse, so h is often measured this way: record the step response, then differentiate it or take its first difference.

For example, let $h(t) = e^{-t}u(t)$. For $t < 0$ the running integral covers no part of h, so $s(t) = 0$. For $t > 0$, integrate from 0 to t with the antiderivative $-e^{-\tau}$:

$$\begin{aligned} s(t) &= \int_{0}^{t} e^{-\tau}\, \mathrm{d}\tau \\ &= \left[-e^{-\tau}\right]_0^t \\ &= -e^{-t} - (-1) = 1 - e^{-t}. \end{aligned}$$

The derivative of $1 - e^{-t}$ is e^{-t}, which returns $h(t)$ for $t > 0$. The final value, 1, is the total area of h.

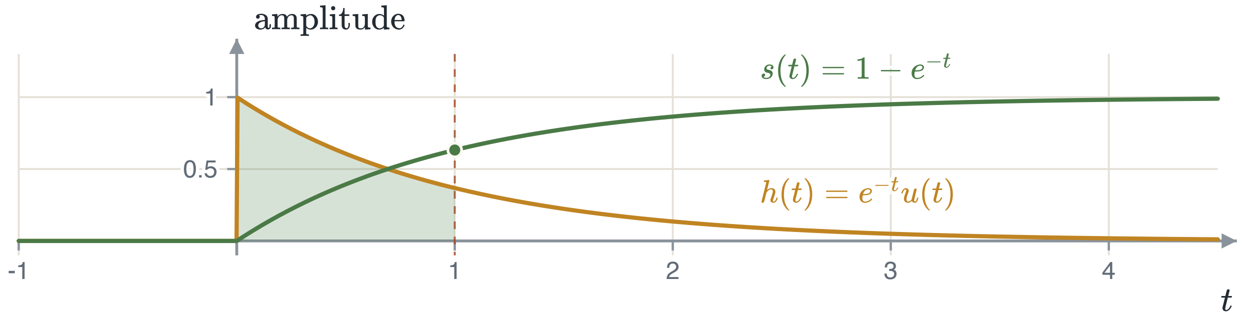

FIGURE 3.20 The shaded area under h from 0 to $t = 1$ is the height of $s(1) = 1 - e^{-1}$.

EXAMPLE 3.6

GIVEN $h[n] = \delta[n] - \delta[n-2]$.

FIND The step response $s[n]$.

METHOD Form the running sum $s[n] = \sum_{k \le n} h[k]$ region by region.

SOLUTION The sum collects the samples of h at $k \le n$. The only non-zero samples are $h[0] = 1$ and $h[2] = -1$.

$$\begin{aligned} n < 0: &\quad s[n] = 0, \\ n = 0: &\quad s[0] = h[0] = 1, \\ n = 1: &\quad s[1] = h[0] + h[1] = 1 + 0 = 1, \\ n \ge 2: &\quad s[n] = h[0] + h[1] + h[2] = 1 + 0 - 1 = 0. \end{aligned}$$

So $s[n] = \delta[n] + \delta[n-1]$.

CHECK Recover h with the first difference: $s[0] - s[-1] = 1 - 0 = 1$, $s[1] - s[0] = 1 - 1 = 0$, $s[2] - s[1] = 0 - 1 = -1$, and 0 after that. This is $\delta[n] - \delta[n-2]$.

The integrator and the differentiator

Two basic operations of calculus are LTI systems. The **integrator** has the output $y(t) = \int_{-\infty}^{t} x(\tau)\,\mathrm{d}\tau$. It is convolution with the unit step, because $u(t-\tau)$ cuts the integral off at $\tau = t$:

$$\begin{aligned} x(t) * u(t) &= \int_{-\infty}^{\infty} x(\tau)\,u(t-\tau)\,\mathrm{d}\tau \\ &= \int_{-\infty}^{t} x(\tau)\,\mathrm{d}\tau. \end{aligned}$$

So the integrator has $h(t) = u(t)$: the running integral of $\delta(t)$ is $u(t)$. The step response $s = h * u$ is therefore h passed through an integrator. The integrator is not BIBO stable, because $\int_{-\infty}^{\infty} |u(t)|\,\mathrm{d}t$ diverges.

The **differentiator** has the output $y(t) = \mathrm{d}x(t)/\mathrm{d}t$. Its impulse response is the output for $x = \delta$, the derivative of the impulse. It is called the **unit doublet**:

$$u_1(t) = \frac{\mathrm{d}\delta(t)}{\mathrm{d}t}.$$

Like δ, the doublet is defined by what it does under convolution, not by its values. Differentiate the sifting form of x with respect to t. Only the factor $\delta(t-\tau)$ depends on t:

$$\begin{aligned} \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,\mathrm{d}\tau \\ &= \int_{-\infty}^{\infty} x(\tau)\,u_1(t-\tau)\,\mathrm{d}\tau \\ &= x(t) * u_1(t). \end{aligned}$$

Put $x = u$ into this rule. The derivative of the step is the impulse, so

$$u(t) * u_1(t) = \frac{\mathrm{d}u(t)}{\mathrm{d}t} = \delta(t).$$

The cascade of an integrator and a differentiator has impulse response δ. The differentiator is the inverse of the integrator, as the first difference is the inverse of the accumulator.

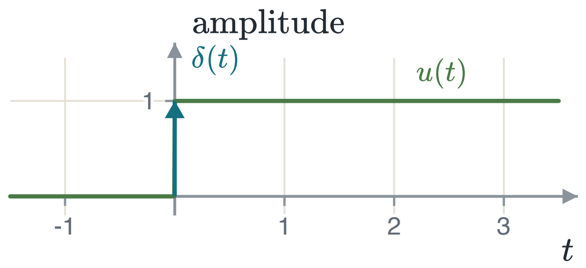

FIGURE 3.21 Integrator: the input $\delta(t)$ gives the output $u(t)$.

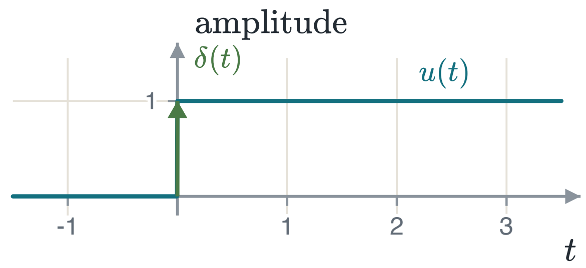

FIGURE 3.22 Differentiator: the input $u(t)$ gives the output $\delta(t)$.

3.5 Difference and differential equations

Many systems are given by an equation that links the output to its own earlier values. This section shows when such an equation describes an LTI system, and how to find its impulse response.

Difference equations and initial rest

A linear constant-coefficient difference equation has the form

DIFFERENCE EQUATION

$$\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k]$$

The coefficients a_k and b_k are constants, and $a_0 \neq 0$.

Move every term except $a_0\, y[n]$ to the right and divide by a_0. This gives a recursion that computes each output from earlier outputs and inputs:

$$y[n] = \frac{1}{a_0}\left(\sum_{k=0}^{M} b_k\, x[n-k] - \sum_{k=1}^{N} a_k\, y[n-k]\right).$$

The recursion needs N earlier outputs before it can start. So the equation alone does not fix the system; a starting condition is also needed.

ⓘ INITIAL REST

If $x[n] = 0$ for $n < n_0$, take $y[n] = 0$ for $n < n_0$. With this condition the system is causal and LTI. A non-zero start would give an output with no input, and a linear system maps the zero input to the zero output.

EXAMPLE 3.7

GIVEN $y[n] = \frac{1}{2}\,y[n-1] + x[n]$, at initial rest.

FIND The impulse response $h[n]$.

METHOD Put $x[n] = \delta[n]$, so $y[n] = h[n]$. Initial rest gives $h[n] = 0$ for $n < 0$. Run the recursion forward from $n = 0$.

SOLUTION The recursion is $h[n] = \frac{1}{2}\,h[n-1] + \delta[n]$. Start from $h[-1] = 0$:

$$\begin{aligned} h[0] &= \tfrac{1}{2}\,h[-1] + \delta[0] = \tfrac{1}{2}\cdot 0 + 1 = 1, \\ h[1] &= \tfrac{1}{2}\,h[0] + \delta[1] = \tfrac{1}{2}\cdot 1 + 0 = \tfrac{1}{2}, \\ h[2] &= \tfrac{1}{2}\,h[1] + \delta[2] = \tfrac{1}{2}\cdot\tfrac{1}{2} + 0 = \tfrac{1}{4}. \end{aligned}$$

For $n \geq 1$ the impulse term is 0, so each step halves the last sample: $h[n] = \frac{1}{2}\,h[n-1]$. Starting from $h[0] = 1$, this gives $h[n] = \left(\frac{1}{2}\right)^n$ for $n \geq 0$. Together, $h[n] = \left(\frac{1}{2}\right)^n u[n]$.

CHECK The formula gives $h[0] = 1$, $h[1] = \frac{1}{2}$ and $h[2] = \frac{1}{4}$, as the recursion did. With the feedback coefficient $-\frac{1}{2}$ instead, $y[n] = -\frac{1}{2}\,y[n-1] + x[n]$ gives $h[0] = 1$, $h[1] = -\frac{1}{2}$ and $h[2] = -\frac{1}{2}\cdot\left(-\frac{1}{2}\right) = \frac{1}{4}$.

Non-recursive and recursive equations

If the equation has no earlier outputs ($N = 0$ and $a_0 = 1$), it is **non-recursive**: $y[n] = \sum_{k=0}^{M} b_k\,x[n-k]$. Put $x = \delta$. Each term $b_k\,\delta[n-k]$ places the value b_k at $n = k$:

$$h[n] = \sum_{k=0}^{M} b_k\,\delta[n-k] = \begin{cases} b_n, & 0 \leq n \leq M \\ 0, & \text{otherwise.} \end{cases}$$

So h has at most $M+1$ non-zero samples. Such a system has a **finite impulse response** (FIR). For example, $y[n] = x[n] + 2\,x[n-1] - x[n-3]$ has $h[n] = \delta[n] + 2\,\delta[n-1] - \delta[n-3]$. Its coefficient b_2 is 0, so $h[2] = 0$ and h has three non-zero samples.

If the equation feeds earlier outputs back, it is **recursive**. Take the first-order case $y[n] = a\,y[n-1] + b\,x[n]$, at rest. As in Example 3.7, $h[0] = a\cdot 0 + b = b$, and for $n \geq 1$ each step multiplies the last sample by a:

$$\begin{aligned} & h[n] = a\,h[n-1], \quad n \geq 1 \\ \Longrightarrow \quad & h[n] = b\,a^n u[n]. \end{aligned}$$

One impulse keeps producing samples for ever. Such a system has an **infinite impulse response** (IIR). It is BIBO stable exactly when $|a| < 1$ (with $b \neq 0$). For $|a| < 1$ the geometric sum gives $\sum_n |h[n]| = |b|\sum_{n\geq 0}|a|^n = |b|/(1-|a|)$, which is finite. For $|a| \geq 1$ every term satisfies $|b|\,|a|^n \geq |b| > 0$, so the terms do not go to zero and the sum diverges.

Two smoothers show the difference. The three-sample average $y[n] = \frac{1}{3}\left(x[n] + x[n-1] + x[n-2]\right)$ is non-recursive, with $h[n] = \frac{1}{3}$ for $n = 0, 1, 2$. The recursive smoother $y[n] = 0.8\,y[n-1] + 0.2\,x[n]$ has $a = 0.8$ and $b = 0.2$, so $h[n] = 0.2\,(0.8)^n u[n]$. Both impulse responses sum to 1: $3\cdot\frac{1}{3} = 1$, and $0.2/(1-0.8) = 1$. So a constant input c gives the output $c\sum_n h[n] = c$ once each smoother has settled.

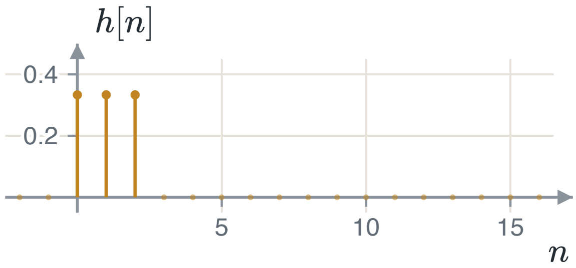

FIGURE 3.23 Non-recursive: $h[n] = \frac{1}{3}\big(\delta[n] + \delta[n-1] + \delta[n-2]\big)$ stops after three samples.

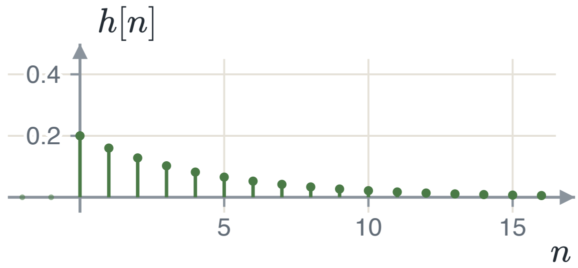

FIGURE 3.24 Recursive: $h[n] = 0.2\,(0.8)^n u[n]$ never stops.

✎ EXAMPLE 3.8

GIVEN A display smooths a sensor reading with $y[n] = 0.8\,y[n-1] + 0.2\,x[n]$, at rest. The reading jumps to 25 °C at $n = 0$: $x[n] = 25\,u[n]$.

FIND The displayed value $y[n]$.

METHOD The system is LTI, so $y = x * h = 25\,(u * h) = 25\,s[n]$, where s is the step response. Find s as the running sum of $h[n] = 0.2\,(0.8)^n u[n]$.

SOLUTION For $n < 0$ no sample of h is included, so $s[n] = 0$. For $n \geq 0$, apply the finite geometric sum with $a = 0.2$, $r = 0.8$, $m = 0$:

$$\begin{aligned} s[n] &= \sum_{k=0}^{n} 0.2\,(0.8)^k \\ &= \frac{0.2\big(1 - 0.8^{\,n+1}\big)}{1 - 0.8} \\ &= 1 - 0.8^{\,n+1}. \end{aligned}$$

So $y[n] = 25\big(1 - 0.8^{\,n+1}\big)u[n]$ °C.

CHECK Run the recursion from rest with $x[n] = 25$: $y[0] = 0.8 \cdot 0 + 0.2 \cdot 25 = 5$, and the formula gives $25(1 - 0.8) = 5$. Next, $y[1] = 0.8 \cdot 5 + 5 = 9$, and $25(1 - 0.64) = 9$. Next, $y[2] = 0.8 \cdot 9 + 5 = 12.2$, and $25(1 - 0.512) = 12.2$. As $n \to \infty$, $0.8^{\,n+1} \to 0$ and the display settles at 25 °C.

Block diagrams

A difference equation can be drawn with three kinds of block: an adder, a gain, and a unit delay D, which outputs its input one sample late. For the first-order equation, the adder output is $y[n]$. The delay holds it for one sample, and the gain a sends it back to the adder:

$$y[n] = \underbrace{b\,x[n]}_{\text{forward path}} + \underbrace{a\,y[n-1]}_{\text{feedback}}.$$

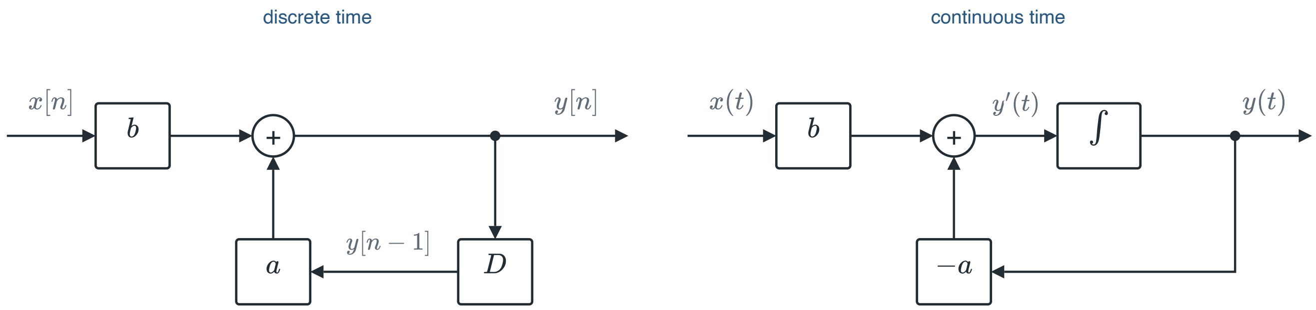

FIGURE 3.25 Left: $y[n] = b\,x[n] + a\,y[n-1]$ with an adder, two gains and a unit delay. Right: $y'(t) = b\,x(t) - a\,y(t)$, where an integrator takes the place of the delay.

Every linear constant-coefficient difference equation can be drawn with adders, gains and unit delays. The diagram is also a program: one pass of the loop computes one sample. With $x[n] = \delta[n]$, $b = 1$ and $a = \frac{1}{2}$, the passes give the impulse response of Example 3.7:

TABLE 3.3 The recursion $y[n] = x[n] + a\,y[n-1]$ computed one sample at a time, with $x[n] = \delta[n]$ and $a = \frac{1}{2}$.

n	$x[n]$	$y[n-1]$	$a\,y[n-1]$	$y[n] = x[n] + a\,y[n-1]$
0	1	0	0	1
1	0	1	$\frac{1}{2}$	$\frac{1}{2}$
2	0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$
3	0	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$
4	0	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{16}$

With $b = 1$ and $a = 1$ the diagram computes $y[n] = y[n-1] + x[n]$. Each new input is added to the total so far, so the system is the accumulator, with $h[n] = 1^n u[n] = u[n]$.

In continuous time an integrator takes the place of the delay. For $y'(t) = b\,x(t) - a\,y(t)$, the adder forms $b\,x - a\,y$, which is y'. The integrator turns y' into y, and the gain $-a$ feeds y back.

A first-order differential equation

The continuous-time counterpart of the first-order recursion is

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} + 2\,y(t) = x(t).$$

Initial rest is defined as in discrete time: if $x(t) = 0$ for $t < t_0$, then $y(t) = 0$ for $t < t_0$. This again makes the system causal and LTI.

The impulse response. Put $x = \delta$, so $h' + 2h = \delta$. For $t < 0$ the input is zero, and at rest $h(t) = 0$. For $t > 0$ the impulse is zero too, so $h' = -2h$. A function whose derivative is -2 times itself is $A\,e^{-2t}$, because $\frac{\mathrm{d}}{\mathrm{d}t}\left(A\,e^{-2t}\right) = -2A\,e^{-2t}$. To find A, integrate both sides of $h' + 2h = \delta$ over a short interval $-\varepsilon < t < \varepsilon$:

$$\begin{aligned}\int_{-\varepsilon}^{\varepsilon} h'(t)\,\mathrm{d}t + 2\int_{-\varepsilon}^{\varepsilon} h(t)\,\mathrm{d}t &= \int_{-\varepsilon}^{\varepsilon} \delta(t)\,\mathrm{d}t \\ h(\varepsilon) - h(-\varepsilon) + 2\int_{-\varepsilon}^{\varepsilon} h(t)\,\mathrm{d}t &= 1.\end{aligned}$$

Here $h(-\varepsilon) = 0$ at rest. The function h contains no impulse: an impulse in h would put a doublet in h', and the right side has none. So h stays finite, and its integral over the shrinking interval goes to zero as $\varepsilon \to 0$. What remains is $h(0^+) = 1$, which gives $A = 1$ and

> IMPULSE RESPONSE
>
> $$h(t) = e^{-2t}u(t)$$

Check it by substitution. Use the product rule, the derivative $u'(t) = \delta(t)$ from Chapter 1, and the sampling property $e^{-2t}\delta(t) = e^0\delta(t) = \delta(t)$:

$$\begin{aligned}\frac{\mathrm{d}h}{\mathrm{d}t} &= -2e^{-2t}u(t) + e^{-2t}\delta(t) \\ &= -2\,h(t) + \delta(t).\end{aligned}$$

So $h' + 2h = \delta$, with $h = 0$ for $t < 0$.

The step response. Integrate h. For $t < 0$, $s(t) = 0$. For $t > 0$, use the antiderivative $-\frac{1}{2}e^{-2\tau}$:

$$\begin{aligned}s(t) &= \int_0^t e^{-2\tau}\,\mathrm{d}\tau \\ &= \left[-\tfrac{1}{2}e^{-2\tau}\right]_0^t \\ &= -\tfrac{1}{2}e^{-2t} + \tfrac{1}{2} = \tfrac{1}{2}\left(1 - e^{-2t}\right).\end{aligned}$$

Check: for $t > 0$, $s' + 2s = e^{-2t} + \left(1 - e^{-2t}\right) = 1$, which is the step input. The output settles at $\frac{1}{2}$.

The same steps with 2 replaced by any $a > 0$ give, for $y' + a\,y = x$ at rest, $h(t) = e^{-at}u(t)$ and $s(t) = \frac{1}{a}\left(1 - e^{-at}\right)u(t)$. A larger a makes h decay faster and makes s settle lower, at $1/a$. For example, $y' + 5\,y = x$ has $h(t) = e^{-5t}u(t)$. With an input gain b, as in $y' + a\,y = b\,x$, linearity multiplies both responses by b. The RC circuit of Section 2.1 has $a = b = 1/RC$, so its impulse response is $h(t) = \frac{1}{RC}\,e^{-t/RC}u(t)$. Chapter 5 finds these systems again from their frequency response.

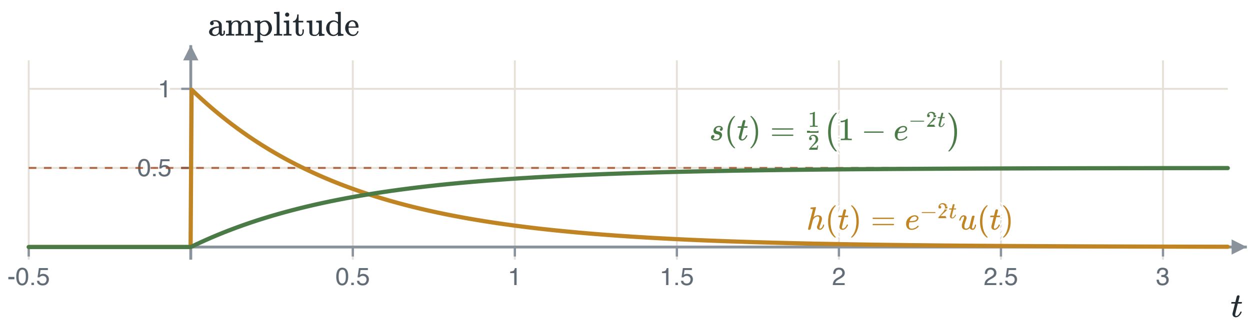

FIGURE 3.26 The impulse response and the step response of $y' + 2y = x$ at rest. The step response settles at $\frac{1}{2}$, the dashed line.

EXAMPLE 3.9

GIVEN A car pulls away against drag: $10\,\dfrac{\mathrm{d}v}{\mathrm{d}t} + v = 30\,u(t)$, with t in seconds and v in m/s, at rest.

FIND The speed $v(t)$.

METHOD Divide by 10 to reach the form $v' + a\,v = x$. The input is then a scaled step, so the output is the same multiple of the step response.

SOLUTION Dividing by 10 gives $v' + 0.1\,v = 3\,u(t)$, so $a = 0.1$. The step response of $v' + 0.1\,v = x$ is $s(t) = \frac{1}{0.1}\big(1 - e^{-0.1t}\big)u(t)$. By linearity the input $3\,u(t)$ gives

$$\begin{aligned} v(t) &= 3\,s(t) \\ &= \frac{3}{0.1}\big(1 - e^{-0.1t}\big)u(t) \\ &= 30\big(1 - e^{-t/10}\big)u(t)\ \mathrm{m/s}. \end{aligned}$$

CHECK Substitute into the original equation for $t > 0$: $10 \cdot \frac{30}{10} e^{-t/10} + 30 - 30\,e^{-t/10} = 30$. The speed starts at 0 and settles at 30 m/s. After one time constant, $v(10) = 30(1 - e^{-1}) \approx 18.96$ m/s.

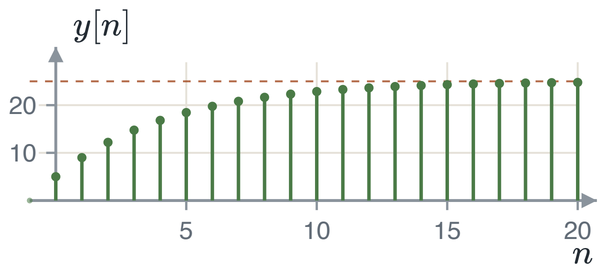

FIGURE 3.27 Example 3.8: $y[n] = 25\big(1 - 0.8^{n+1}\big)u[n]$ settles at 25 °C.

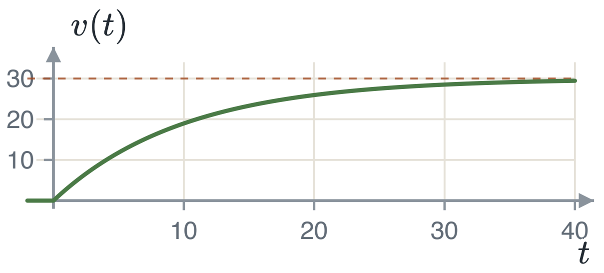

FIGURE 3.28 Example 3.9: $v(t) = 30\big(1 - e^{-t/10}\big)$ m/s settles at 30 m/s.

3.6 Summary

TABLE 3.4 The results of Chapter 3.

RESULT	STATEMENT
Impulse response	The output h when the input is $\delta[n]$, or $\delta(t)$.
Representation	$x[n] = \sum_k x[k]\,\delta[n-k]$, and $x(t) = \int x(\tau)\,\delta(t-\tau)\,\mathrm{d}\tau$.
Convolution	$y[n] = \sum_k x[k]\,h[n-k]$ and $y(t) = \int x(\tau)\,h(t-\tau)\,\mathrm{d}\tau$, for LTI systems only.
Building $h[n-k]$	Flip, then shift. The sample $h[0]$ sits at $k=n$. Without the flip the sum is a correlation.
Three checks	Continuity at each boundary. Supports add. Areas, or sums, multiply.
Interconnections	Parallel: $h_1 + h_2$. Cascade: $h_1 * h_2$, in either order, for LTI systems only.
Memoryless, causal	Memoryless if and only if $h = a\,\delta$. Causal if and only if $h = 0$ for negative time.
BIBO stable	If and only if $\sum_k \lvert h[k]\rvert < \infty$, or $\int \lvert h(t)\rvert\,\mathrm{d}t < \infty$.
Step response	$s = h * u$, the running sum or running integral of h. Back to h: $h[n] = s[n] - s[n-1]$ and $h(t) = s'(t)$.
Difference equations	At initial rest the system is causal and LTI. $y[n] = a\,y[n-1] + x[n]$ has $h[n] = a^n u[n]$.

WHERE CHAPTER 4 BEGINS

Put $x(t) = e^{j\omega t}$ into the second form of the convolution integral:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)\, e^{j\omega(t-\tau)}\,\mathrm{d}\tau \\ &= e^{j\omega t}\int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\,\mathrm{d}\tau. \end{aligned}$$

The integral does not depend on t. The output is the input times one number, so a complex exponential passes through an LTI system unchanged in form.

Exercises

3.1 Compute $\{1,2,3\} * \{1,-1\}$ with both sequences starting at $n=0$, and check your answer with the sum rule.
Answer: $\{1,1,1,-3\}$; the sums multiply as $6 \times 0 = 0$.

3.2 Let $h[n] = \left(\frac{1}{3}\right)^n u[n]$. Is the system stable? Is it causal? Is it memoryless?
Answer: Stable ($\sum \lvert h\rvert = 3/2$), causal, not memoryless.

3.3 Find $y(t) = u(t) * u(t)$ and sketch it. This is the step response of the integrator.
Answer: $y(t) = t\,u(t)$.

3.4 Two systems with $h_1[n] = \delta[n] + \delta[n-1]$ and $h_2[n] = \delta[n] - \delta[n-1]$ are cascaded. Find the impulse response of the cascade.
Answer: $\delta[n] - \delta[n-2]$.

3.5 $x(t) = 1$ on $0 < t < 3$ and $h(t) = t$ on $0 < t < 2$. List the case boundaries before computing anything, then find $y(t)$. The window is now wider than the ramp, which changes the middle case.
Answer: Boundaries at $t = 0, 2, 3, 5$; five cases. $y = 0$, $\frac{1}{2}t^2$, 2, $2 - \frac{1}{2}(t-3)^2$, 0.

3.6 A causal LTI system has the step response $s(t) = \left(1 - e^{-3t}\right)u(t)$. Find $h(t)$.
Answer: $h(t) = 3\,e^{-3t}u(t)$.

3.7 Find the impulse response of $y[n] = 0.9\,y[n-1] + x[n]$ at initial rest. Is the system BIBO stable?
Answer: $h[n] = 0.9^{\,n}u[n]$. Stable: $\sum_n |h[n]| = 10$.

CHAPTER

4

Fourier series

Fourier series are used to find the response of a linear time-invariant (LTI) system to a periodic signal. An LTI system returns each complex exponential in the same form and multiplies it by one complex number. Once a periodic signal is written as a sum of these exponentials, convolution becomes one multiplication for each harmonic. The chapter has two central equations: the synthesis equation, which builds the signal from its coefficients, and the analysis equation, which finds the coefficients from the signal.

THE CHAPTER IN TWO LINES

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{jk\omega_0 t} \qquad \text{(synthesis)}$$

$$a_k = \frac{1}{T_0} \int_{T_0} x(t) \, e^{-jk\omega_0 t} \, \mathrm{d}t \qquad \text{(analysis)}$$

The frequency response of a system states how it changes each frequency. Multiply each input coefficient by the frequency response at its harmonic to obtain the output coefficient.

4.1 The eigenfunction property

This section shows why complex exponentials are the building blocks of LTI analysis. The result is short, and the rest of the chapter depends on it.

DEFINITION

A signal is an **eigenfunction** of a system when the output is that same signal multiplied by a constant. The constant is the **eigenvalue**.

Continuous time

Put $x(t) = e^{st}$, with s any complex number, into the convolution integral, whose limits are $-\infty$ and ∞. Then split the exponential with $e^{s(t-\tau)} = e^{st}e^{-s\tau}$. The factor e^{st} does not depend on τ, so it moves outside the integral.

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, \mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, \mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} h(\tau)\, e^{st} e^{-s\tau}\, \mathrm{d}\tau \\
&= e^{st} \underbrace{\int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, \mathrm{d}\tau}_{H(s)}.
\end{aligned}
$$

The integral that remains does not depend on t. It is one complex number for each value of s. We call it $H(s)$, the eigenvalue of the system for the eigenfunction e^{st}.

Discrete time

The same steps apply in discrete time. Put $x[n] = z^n$, with z any complex number, into the convolution sum and use $z^{n-k} = z^n z^{-k}$. The factor z^n does not depend on k, so it moves outside the sum.

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] \\
&= \sum_{k=-\infty}^{\infty} h[k]\, z^{n-k} \\
&= \sum_{k=-\infty}^{\infty} h[k]\, z^{n} z^{-k} \\
&= z^n \underbrace{\sum_{k=-\infty}^{\infty} h[k]\, z^{-k}}_{H(z)}.
\end{aligned}
$$

EIGENFUNCTION PROPERTY

$$
e^{st} \;\to\; h(t) \;\to\; H(s)\, e^{st}, \qquad H(s) = \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, \mathrm{d}\tau
$$

$$
z^{n} \;\to\; h[n] \;\to\; H(z)\, z^{n}, \qquad H(z) = \sum_{k=-\infty}^{\infty} h[k]\, z^{-k}
$$

Keep the two statements apart. The discrete-time output is a sequence, $y[n] = H(z)\, z^n$. A discrete-time line contains no t and no s.

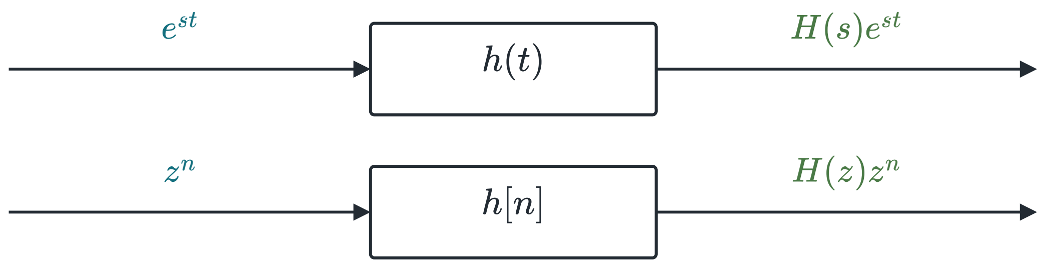

FIGURE 4.1 The continuous-time and the discrete-time statements, side by side.

Set $s = j\omega$ and $z = e^{j\omega}$ with ω real. This gives the two **frequency responses**, $H(j\omega)$ in rad/s and $H(e^{j\omega})$ in rad/sample. Each one states what the system does to the complex exponential of frequency ω. The discrete-time frequency response repeats every 2π in ω, because k is an integer and so $e^{-j2\pi k} = 1$:

$$\begin{aligned} H\left(e^{j(\omega+2\pi)}\right) &= \sum_{k=-\infty}^{\infty} h[k]\, e^{-j(\omega+2\pi)k} \\ &= \sum_{k=-\infty}^{\infty} h[k]\, e^{-j\omega k}\, e^{-j2\pi k} \\ &= \sum_{k=-\infty}^{\infty} h[k]\, e^{-j\omega k} = H(e^{j\omega}). \end{aligned}$$

The continuous-time frequency response has no such rule, because τ is not restricted to integers.

EXAMPLE 4.1 – A COSINE THROUGH $h(t) = e^{-t}u(t)$

GIVEN An LTI system with $h(t) = e^{-t}u(t)$ and the input $x(t) = \cos 2t$.

FIND The output, and whether $\cos 2t$ is an eigenfunction of this system.

METHOD A cosine is not itself a complex exponential. Write it as two exponentials with Euler's relation, $\cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$. Each exponential is an eigenfunction, so multiply it by its own eigenvalue and add the two results.

SOLUTION First the frequency response. Substitute h into the definition. The step $u(\tau)$ makes the lower limit zero, and the two exponentials combine into one:

$$\begin{aligned} H(j\omega) &= \int_{-\infty}^{\infty} e^{-\tau}u(\tau)\, e^{-j\omega\tau}\, \mathrm{d}\tau = \int_{0}^{\infty} e^{-(1+j\omega)\tau}\, \mathrm{d}\tau \\ &= \left[\frac{e^{-(1+j\omega)\tau}}{-(1+j\omega)}\right]_0^{\infty} = \frac{0-1}{-(1+j\omega)} = \frac{1}{1+j\omega}. \end{aligned}$$

The upper limit gives zero because $|e^{-(1+j\omega)\tau}| = e^{-\tau} \to 0$. At $\omega = 2$ rad/s, $H(j2) = 1/(1+j2)$. Its magnitude is $1/\sqrt{1+2^2} = 1/\sqrt{5} = 0.447$ and its phase is $-\arctan 2 = -1.107$ rad. At $\omega = -2$, $H(-j2) = 1/(1-j2)$, which is the conjugate: $0.447\, e^{+j1.107}$. Now send each half of the cosine through the system:

$$\begin{aligned} \cos 2t &= \tfrac{1}{2}e^{j2t} + \tfrac{1}{2}e^{-j2t} \\ &\to \tfrac{1}{2}H(j2)\, e^{j2t} + \tfrac{1}{2}H(-j2)\, e^{-j2t} \\ &= \tfrac{1}{2}(0.447)\, e^{j(2t-1.107)} + \tfrac{1}{2}(0.447)\, e^{-j(2t-1.107)} \\ &= 0.447\cos(2t - 1.107). \end{aligned}$$

The last line uses $e^{j\theta} + e^{-j\theta} = 2\cos\theta$. The output has the input frequency. Only the amplitude and the phase change. The output is not a constant times $\cos 2t$, because of the phase -1.107 rad. So $\cos 2t$ is not an eigenfunction of this system. The two exponentials e^{j2t} and e^{-j2t} are.

CHECK Evaluate the convolution directly at $t = 0$: $y(0) = \int_0^{\infty} e^{-\tau}\cos(-2\tau)\,\mathrm{d}\tau = \mathrm{Re}\int_0^{\infty} e^{-(1-j2)\tau}\,\mathrm{d}\tau = \mathrm{Re}\,\frac{1}{1-j2} = \mathrm{Re}\,\frac{1+j2}{5} = 0.2$. The formula gives

$$0.447\cos(-1.107) = 0.447 \times 0.447 = 0.2, \text{ since } \cos(\arctan 2) = 1/\sqrt{5}.$$

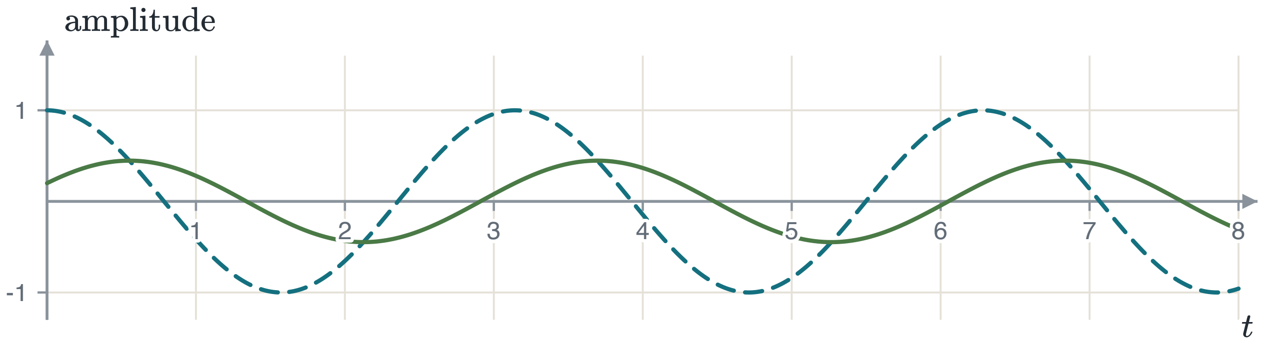

FIGURE 4.2 The input $\cos 2t$ (dashed) and the output $0.447\cos(2t - 1.107)$ (solid) of Example 4.1. The frequency stays; the amplitude and the phase change.

EXAMPLE 4.2 — A COSINE SEQUENCE THROUGH THE TWO-POINT AVERAGE

GIVEN The system $y[n] = \frac{1}{2}x[n] + \frac{1}{2}x[n-1]$, with the inputs $x[n] = \cos(2\pi n/3)$ and $x[n] = (-1)^n$.

FIND Both outputs.

METHOD The impulse response is $h[n] = \frac{1}{2}\delta[n] + \frac{1}{2}\delta[n-1]$. Find $H(e^{j\omega})$ from the definition. Write the cosine as two exponentials. Write $(-1)^n$ as z^n with $z = -1$.

SOLUTION Only $k = 0$ and $k = 1$ contribute to the sum. Factor out $e^{-j\omega/2}$ and use $e^{j\theta} + e^{-j\theta} = 2\cos\theta$:

$$\begin{aligned} H(e^{j\omega}) &= \sum_{k=-\infty}^{\infty} h[k]\, e^{-j\omega k} = \tfrac{1}{2} + \tfrac{1}{2}e^{-j\omega} \\ &= \tfrac{1}{2}e^{-j\omega/2}\left(e^{j\omega/2} + e^{-j\omega/2}\right) = e^{-j\omega/2}\cos(\omega/2). \end{aligned}$$

At $\omega = 2\pi/3$ rad/sample, $\cos(\pi/3) = \frac{1}{2}$, so $H(e^{j2\pi/3}) = \frac{1}{2}e^{-j\pi/3}$ and $H(e^{-j2\pi/3}) = \frac{1}{2}e^{+j\pi/3}$. Then

$$\begin{aligned} \cos\left(\tfrac{2\pi n}{3}\right) &= \tfrac{1}{2}e^{j2\pi n/3} + \tfrac{1}{2}e^{-j2\pi n/3} \\ &\to \tfrac{1}{2}\cdot\tfrac{1}{2}e^{-j\pi/3}e^{j2\pi n/3} + \tfrac{1}{2}\cdot\tfrac{1}{2}e^{j\pi/3}e^{-j2\pi n/3} \\ &= \tfrac{1}{2}\cos\left(\tfrac{2\pi n}{3} - \tfrac{\pi}{3}\right). \end{aligned}$$

For the second input, $(-1)^n = z^n$ with $z = -1$. The eigenvalue is $H(-1) = \frac{1}{2} + \frac{1}{2}(-1)^{-1} = \frac{1}{2} - \frac{1}{2} = 0$, so $y[n] = 0$ for every n.

CHECK Apply the rule directly at $n = 0$: $y[0] = \frac{1}{2}\cos 0 + \frac{1}{2}\cos(-2\pi/3) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$. The formula gives $\frac{1}{2}\cos(-\pi/3) = \frac{1}{4}$. For $(-1)^n$ the rule gives $\frac{1}{2}(-1)^n + \frac{1}{2}(-1)^{n-1} = 0$.

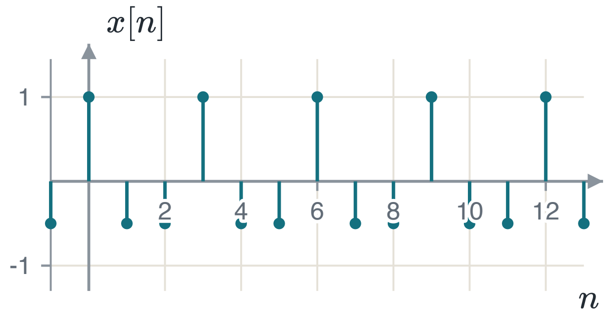

FIGURE 4.3 The input $x[n] = \cos(2\pi n/3)$.

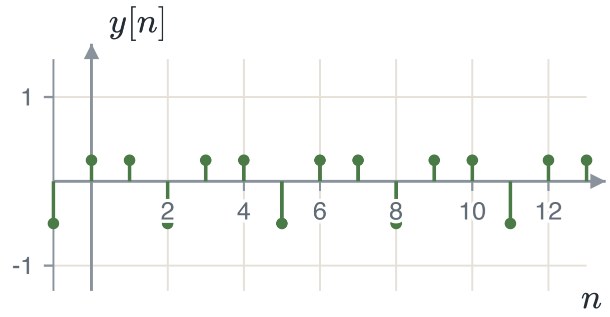

FIGURE 4.4 The output $\frac{1}{2}\cos(2\pi n/3 - \pi/3)$: the same frequency.

A sum of eigenfunctions

Now let the input be a sum of complex exponentials, $x(t) = \sum_k a_k e^{s_k t}$. The system is linear, so each term passes through on its own and the outputs add. The eigenfunction property multiplies term k by $H(s_k)$:

> SUPERPOSITION OF EIGENFUNCTIONS
>
> $$x(t) = \sum_k a_k\, e^{s_k t} \;\longrightarrow\; y(t) = \sum_k a_k\, H(s_k)\, e^{s_k t}$$
>
> $$x[n] = \sum_k a_k\, z_k^n \;\longrightarrow\; y[n] = \sum_k a_k\, H(z_k)\, z_k^n$$
>
> No convolution is needed. The output needs one multiplication for each term. For example, $x(t) = 2e^{jt} + e^{j3t}$ with $H(j1) = 0.5$ and $H(j3) = 0$ gives $y(t) = 2(0.5)\, e^{jt} + 1(0)\, e^{j3t} = e^{jt}$.

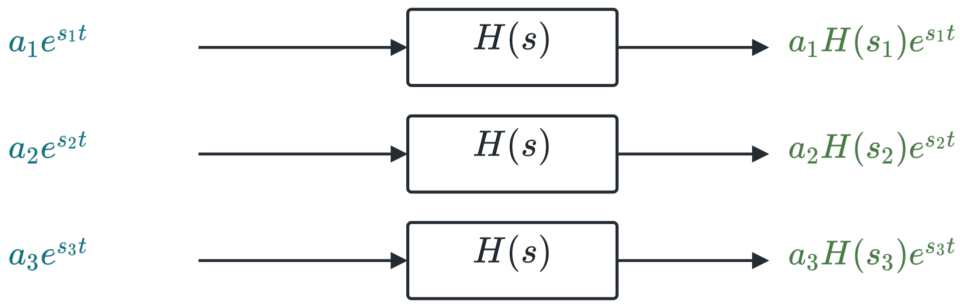

FIGURE 4.5 Each exponential passes through the system on its own. Superposition then adds the three outputs.

EXAMPLE 4.3 — A PURE DELAY

GIVEN An LTI system with $y(t) = x(t-3)$.

FIND The output for $x(t) = e^{j2t}$, and for $x(t) = \cos 4t + \cos 7t$, using eigenfunctions only.

METHOD The inputs are complex exponentials or sums of them, so the eigenfunction method applies. Write $h(t)$ from the rule, find $H(s)$ from the definition, and multiply each exponential by its own eigenvalue.

SOLUTION A delay by 3 s is convolution with $\delta(t-3)$, because $x(t) * \delta(t-3) = x(t-3)$. So $h(t) = \delta(t-3)$. Put this into the definition of $H(s)$. The sifting property replaces τ by 3 in the exponential:

$$H(s) = \int_{-\infty}^{\infty} \delta(\tau - 3)\, e^{-s\tau}\, \mathrm{d}\tau = e^{-3s}.$$

For $x(t) = e^{j2t}$ the exponent gives $s = j2$, so

$$y(t) = H(j2)\, e^{j2t} = e^{-j6} e^{j2t} = e^{j2(t-3)}.$$

For the cosines, first write each one as two exponentials with Euler's relation. Then multiply each exponential by its own eigenvalue. For $\cos 4t$ the two values of s are $\pm j4$, with $H(j4) = e^{-j12}$ and $H(-j4) = e^{+j12}$:

$$\begin{aligned}\cos 4t &= \tfrac{1}{2}e^{j4t} + \tfrac{1}{2}e^{-j4t} \\ &\to \tfrac{1}{2}e^{-j12}e^{j4t} + \tfrac{1}{2}e^{j12}e^{-j4t} \\ &= \tfrac{1}{2}e^{j4(t-3)} + \tfrac{1}{2}e^{-j4(t-3)} \\ &= \cos\big(4(t-3)\big).\end{aligned}$$

The same steps with $s = \pm j7$ and $H(\pm j7) = e^{\mp j21}$ give $\cos\big(7(t-3)\big)$. The system is linear, so the two outputs add:

$$y(t) = \cos\big(4(t-3)\big) + \cos\big(7(t-3)\big).$$

CHECK

Apply $y(t) = x(t-3)$ directly to both inputs. The answers agree. Note also that $|H(j\omega)| = |e^{-j3\omega}| = 1$ and $\angle H(j\omega) = -3\omega$. A delay changes no amplitude. It adds a phase proportional to frequency: at $\omega = 2$ rad/s the phase lag is $2 \times 3 = 6$ rad.

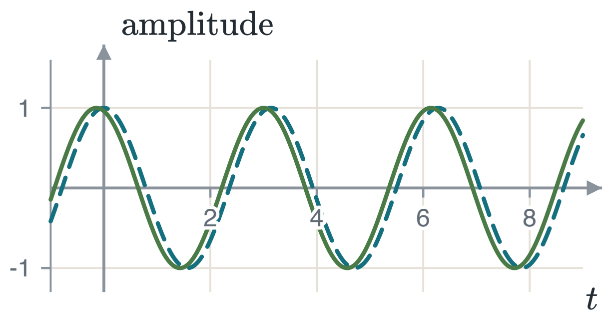

FIGURE 4.6 $\cos 2t$ (dashed) and $\cos\big(2(t-3)\big)$ (solid).

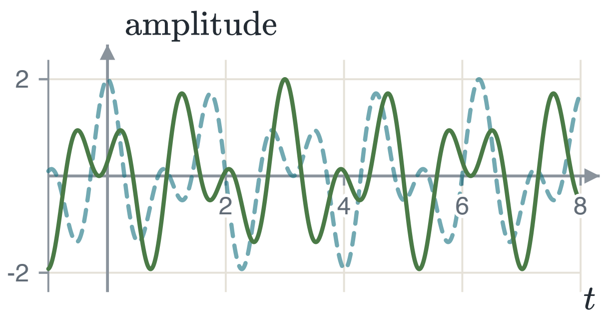

FIGURE 4.7 $x(t) = \cos 4t + \cos 7t$ (dashed) and the delayed output (solid).

The next task is to write a signal as a sum of complex exponentials. For a periodic signal, the rest of this chapter does exactly that. A sinusoid is the one input whose form an LTI system keeps, so one complex number at each frequency describes the whole system.

4.2 Synthesis and analysis

Which periodic signals have a series

Not every periodic signal can be written as a sum of complex exponentials. The following three conditions guarantee that it can. Let $x(t) = x(t + T_0)$ for every t.

1. Over one period, x is absolutely integrable: $\int_{T_0} |x(t)|\, \mathrm{d}t < \infty$.
2. In any finite interval, x has only a finite number of maxima and minima.
3. In any finite interval, x has only a finite number of jumps, and each jump is finite.

> SUFFICIENT, NOT NECESSARY
>
> These are the **Dirichlet conditions**. A signal that meets all three has a Fourier series. The series equals $x(t)$ wherever x is continuous. At a jump it equals the midpoint of the jump. A signal that fails one condition may still have a series: the conditions guarantee, they do not exclude.

The three conditions test different things. Each signal below has period 1 or 4, meets the earlier conditions and fails one. The signal $x(t) = 1/t$ for $0 < t \leq 1$ fails the first, because $\int_0^1 \mathrm{d}t/t$ diverges. The signal $\sin(2\pi/t)$ for $0 < t \leq 1$ has $|x| \leq 1$, so its area is finite, but it oscillates infinitely often near $t = 0$: it fails only the second. A staircase whose steps halve in height and in width fails only the third. Its area over one period is $2 + \frac{1}{2} + \frac{1}{8} + \cdots = 2/(1 - \frac{1}{4}) = 8/3$, which is finite, but it has infinitely many jumps. Every signal in this chapter, including square waves, sawtooths, impulse trains and sums of sinusoids, meets all three conditions.

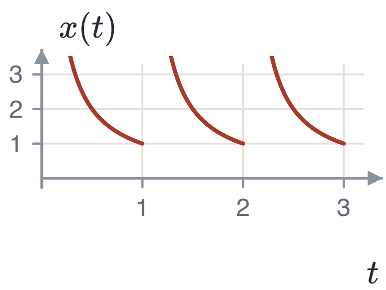

FIGURE 4.8 $1/t$: condition 1 fails.

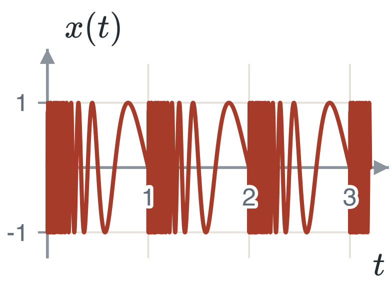

FIGURE 4.9 $\sin(2\pi/t)$: condition 2 fails.

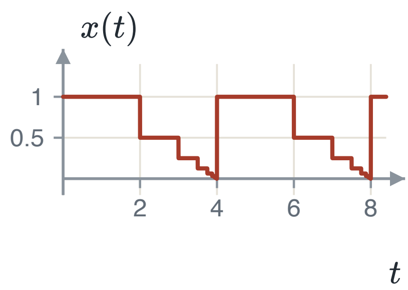

FIGURE 4.10 Halving steps: condition 3 fails.

The synthesis equation

> SYNTHESIS EQUATION
>
> $$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}, \qquad \omega_0 = \frac{2\pi}{T_0}$$
>
> T_0 is the fundamental period in seconds and ω_0 is the fundamental frequency in rad/s. The term $e^{jk\omega_0 t}$ is the k-th **harmonic**. The complex numbers a_k are the **Fourier series coefficients**: $|a_k|$ is the amplitude of the k-th harmonic and $\angle a_k$ is its phase.

Every harmonic repeats with the period T_0, so the sum does too. Replace t by $t + T_0$ and use $\omega_0 T_0 = 2\pi$:

$$e^{jk\omega_0(t+T_0)} = e^{jk\omega_0 t}\, e^{jk\omega_0 T_0} = e^{jk\omega_0 t}\, e^{jk2\pi} = e^{jk\omega_0 t}.$$

When a signal is already a sum of sinusoids, the coefficients can be read off. Find T_0, then $\omega_0 = 2\pi/T_0$. Write each sinusoid with Euler's relations, $\cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$ and $\sin\theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$. Match each exponent with $jk\omega_0 t$ and read off k. The first step needs the period of a sum.

Fundamental period of a sum

A sum of periodic signals is periodic when each component period is a rational multiple of every other. Its period T_0 is the smallest positive time that is a whole number of each component period. Write each component period as a fraction p_i/q_i in lowest terms. A constant term has no period and is left out.

PERIOD OF A SUM

$$T_0 = \mathrm{LCM}\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \dots\right) = \frac{\mathrm{LCM}(p_1, p_2, \dots)}{\mathrm{GCD}(q_1, q_2, \dots)}$$

The result is a least common multiple of fractions, so it need not be an integer.

The rule follows in two steps. Write $L = \mathrm{LCM}(p_1, p_2, \dots)$ and $G = \mathrm{GCD}(q_1, q_2, \dots)$. First, L/G is a common period. Divide it by the period p_i/q_i:

$$\frac{L/G}{p_i/q_i} = \frac{L}{p_i} \cdot \frac{q_i}{G}.$$

Both factors are whole numbers, because p_i divides L and G divides q_i. Second, no common period is smaller. Let $T = r/s$, in lowest terms, be any common period. Then $T/(p_i/q_i) = rq_i/(sp_i)$ is a whole number for every i. The number p_i shares no factor with q_i, so p_i divides r. This holds for every i, so L divides r and $r \geq L$. The number s shares no factor with r, so s divides q_i. This holds for every i, so s divides G and $s \leq G$. Therefore $T = r/s \geq L/G$.

EXAMPLE 4.4 — FOUR FUNDAMENTAL PERIODS

GIVEN Sums of periodic components with these periods, in seconds: (a) 1 and $\frac{2}{3}$; (b) $\frac{3}{5}$ and $\frac{8}{5}$; (c) 2, 1 and $\frac{2}{3}$; (d) $\frac{2}{9}$ and $\frac{8}{21}$.

FIND The fundamental period of each sum.

METHOD Write each period in lowest terms. Divide the least common multiple of the numerators by the greatest common divisor of the denominators. Then divide T_0 by each period.

SOLUTION

$$\begin{aligned}
&\text{(a)}\quad \tfrac{1}{1},\ \tfrac{2}{3}: && T_0 = \frac{\mathrm{LCM}(1,2)}{\mathrm{GCD}(1,3)} = \frac{2}{1} = 2\ \text{s},\\
&\text{(b)}\quad \tfrac{3}{5},\ \tfrac{8}{5}: && T_0 = \frac{\mathrm{LCM}(3,8)}{\mathrm{GCD}(5,5)} = \frac{24}{5}\ \text{s},\\
&\text{(c)}\quad \tfrac{2}{1},\ \tfrac{1}{1},\ \tfrac{2}{3}: && T_0 = \frac{\mathrm{LCM}(2,1,2)}{\mathrm{GCD}(1,1,3)} = \frac{2}{1} = 2\ \text{s},\\
&\text{(d)}\quad \tfrac{2}{9},\ \tfrac{8}{21}: && T_0 = \frac{\mathrm{LCM}(2,8)}{\mathrm{GCD}(9,21)} = \frac{8}{3}\ \text{s}.
\end{aligned}$$

In (d), $\mathrm{GCD}(9,21) = 3$ because $9 = 3^2$ and $21 = 3 \cdot 7$.

CHECK Divide T_0 by each period. (a) $2 \div 1 = 2$ and $2 \div \frac{2}{3} = 3$. (b) $\frac{24}{5} \div \frac{3}{5} = 8$ and $\frac{24}{5} \div \frac{8}{5} = 3$. (c) 1, 2 and 3. (d) $\frac{8}{3} \div \frac{2}{9} = 12$ and $\frac{8}{3} \div \frac{8}{21} = 7$. Every quotient is a whole number. In each case the quotients share no common factor, so T_0 divided by any whole number greater than 1 is no longer a common period.

ⓧ DIVIDE BY THE GCD

For $\frac{2}{9}$ and $\frac{8}{21}$, dividing by $\mathrm{LCM}(9,21) = 63$ instead of the GCD gives $\frac{8}{63}$. It is not a multiple of either period: $\frac{8}{63} \div \frac{2}{9} = \frac{4}{7}$. Always divide the answer by each period and confirm that the results are whole numbers.

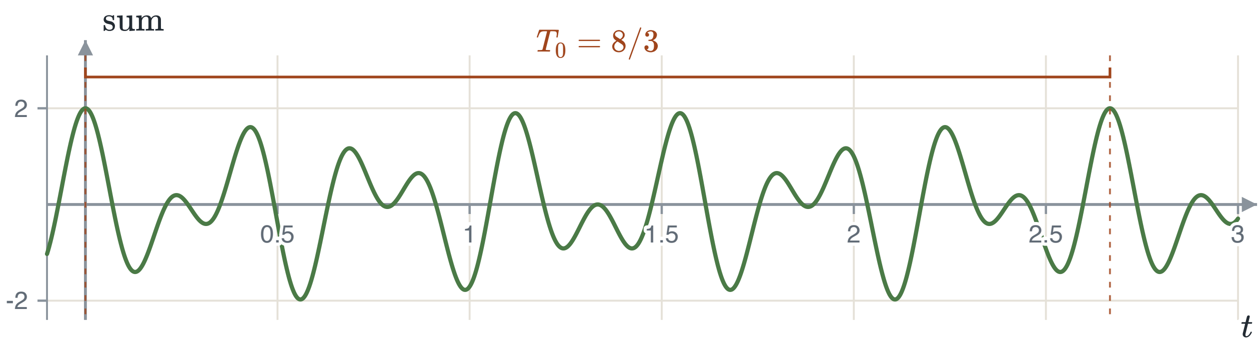

FIGURE 4.11 Two cosines of periods $2/9$ s and $8/21$ s, added. The sum first repeats after $8/3$ s: twelve cycles of the first and seven of the second.

EXAMPLE 4.5 — READING OFF THE COEFFICIENTS

GIVEN $x(t) = 1 + \frac{1}{2}\cos(2\pi t) + \sin(3\pi t)$.

FIND Every Fourier series coefficient, with its magnitude and phase.

METHOD Find T_0, hence ω_0. Expand with Euler's relations and match each exponent with $jk\omega_0 t$.

SOLUTION First the period. $\cos(2\pi t)$ repeats every $2\pi/(2\pi) = 1$ s and $\sin(3\pi t)$ every $2\pi/(3\pi) = 2/3$ s. This is case (a) of Example 4.4, so $T_0 = 2$ s and $\omega_0 = 2\pi/T_0 = \pi$ rad/s. Now expand each sinusoid with Euler's relations:

$$\begin{aligned}
\tfrac{1}{2}\cos(2\pi t) &= \tfrac{1}{2}\cdot\tfrac{1}{2}\left(e^{j2\pi t} + e^{-j2\pi t}\right) = \tfrac{1}{4}e^{j2\pi t} + \tfrac{1}{4}e^{-j2\pi t},\\
\sin(3\pi t) &= \tfrac{1}{2j}\left(e^{j3\pi t} - e^{-j3\pi t}\right) = \tfrac{1}{2j}e^{j3\pi t} - \tfrac{1}{2j}e^{-j3\pi t}.
\end{aligned}$$

Match each exponent with $jk\omega_0 t = jk\pi t$. Here $2\pi t = 2\omega_0 t$, so $k = \pm 2$, and $3\pi t = 3\omega_0 t$, so $k = \pm 3$. The constant is $1 \cdot e^{j0\omega_0 t}$, so $k = 0$. Collecting the terms,

$$\begin{aligned} x(t) = \underbrace{1}_{k=0} &+ \underbrace{\tfrac{1}{4}e^{j2\omega_0 t} + \tfrac{1}{4}e^{-j2\omega_0 t}}_{k=\pm 2} \\ &+ \underbrace{\tfrac{1}{2j}e^{j3\omega_0 t} - \tfrac{1}{2j}e^{-j3\omega_0 t}}_{k=\pm 3}, \end{aligned}$$

so $a_0 = 1$, $a_{\pm 2} = \frac{1}{4}$, $a_3 = \frac{1}{2j}$, $a_{-3} = -\frac{1}{2j}$, and every other coefficient is zero.

CHECK $a_0 = 1$ is also the average of the signal: both sinusoids complete whole cycles in one period and average to zero, which leaves the constant. Magnitudes: $|a_0| = 1$, $|a_{\pm 2}| = \frac{1}{4}$, $|a_{\pm 3}| = \frac{1}{2}$. Phases: $\frac{1}{2j} = -\frac{j}{2}$, so $\angle a_3 = -\pi/2$, and $a_{-3} = +\frac{j}{2}$, so $\angle a_{-3} = +\pi/2$. The signal is real, so the magnitudes must be even in k and the phases odd, and they are.

⊗ THE CONSTANT IS A HARMONIC

The constant 1 is the $k = 0$ term, $1 \cdot e^{j0\omega_0 t}$. So $a_0 = 1$, not 0. It is the average of x over one period.

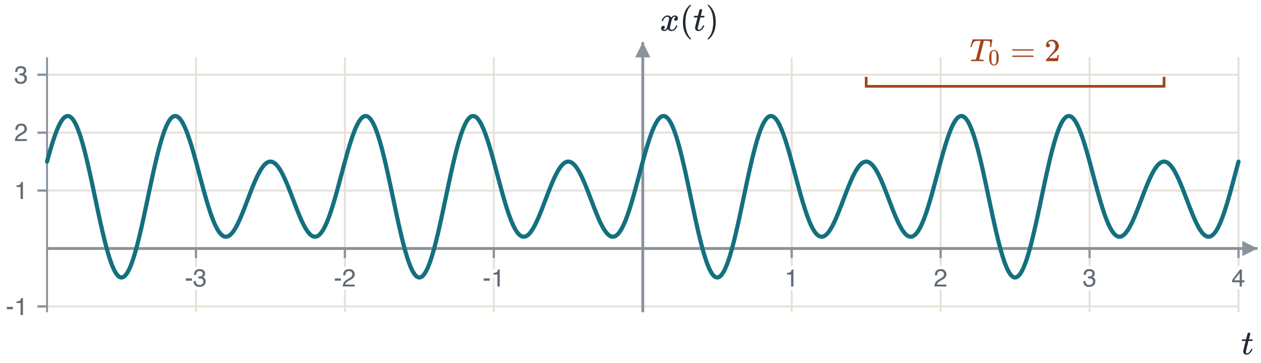

FIGURE 4.12 $x(t) = 1 + \frac{1}{2}\cos(2\pi t) + \sin(3\pi t)$: a constant and two sinusoids, with period $T_0 = 2$ s.

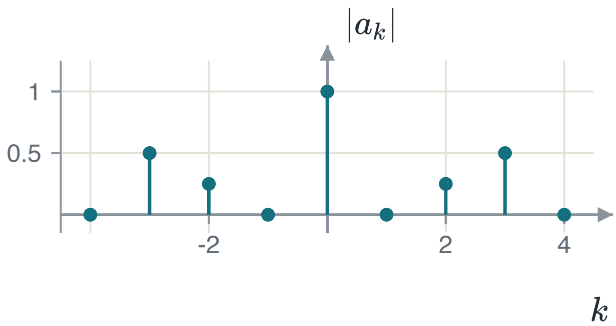

FIGURE 4.13 Magnitudes, even in k.

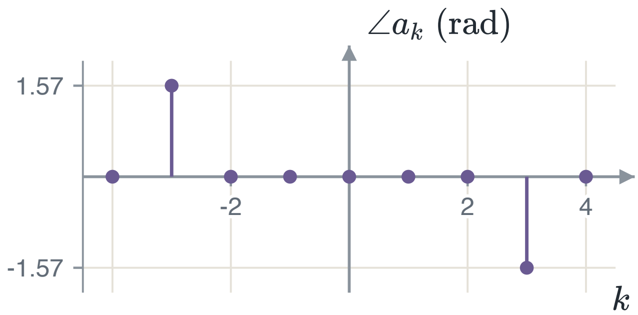

FIGURE 4.14 Phases, odd in k. A value of ± 1.57 rad is $\pm\pi/2$.

The analysis equation

Reading coefficients off works only for a sum of sinusoids. Every other periodic signal needs a formula.

ANALYSIS EQUATION

$$a_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t}\,\mathrm{d}t, \qquad \omega_0 = \frac{2\pi}{T_0}$$

$\int_{T_0}$ means an integral over any one period. The integrand $x(t)e^{-jk\omega_0 t}$ is a product of two factors of period T_0, so it repeats with period T_0, and every interval of length T_0 gives the same value. Synthesis uses $e^{+jk\omega_0 t}$ and analysis uses $e^{-jk\omega_0 t}$, with the same ω_0. The sign of the exponent is the only thing that separates the pair.

To prove the analysis equation, isolate one coefficient, a_n. Multiply both sides of the synthesis equation by $e^{-jn\omega_0 t}$ and integrate over one period. The sum and the integral can be exchanged, and a_k does not depend on t, so it moves outside the integral. The two exponentials combine because $e^{jk\omega_0 t}e^{-jn\omega_0 t} = e^{j(k-n)\omega_0 t}$.

$$\begin{aligned}\int_{T_0} x(t)\, e^{-jn\omega_0 t}\,\mathrm{d}t &= \int_{T_0}\Bigl(\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\Bigr)e^{-jn\omega_0 t}\,\mathrm{d}t \\ &= \sum_{k=-\infty}^{\infty} a_k \int_{T_0} e^{j(k-n)\omega_0 t}\,\mathrm{d}t.\end{aligned}$$

Now evaluate the inner integral from $-T_0/2$ to $T_0/2$. Put $m = k - n$, an integer, and take $m \neq 0$ first, so that the antiderivative may divide by m. At the limits $\omega_0 T_0/2 = \pi$. The difference of the two exponentials is $2j$ times a sine, by $e^{j\theta} - e^{-j\theta} = 2j\sin\theta$.

$$\begin{aligned}\int_{-T_0/2}^{T_0/2} e^{jm\omega_0 t}\,\mathrm{d}t &= \left[\frac{e^{jm\omega_0 t}}{jm\omega_0}\right]_{-T_0/2}^{T_0/2} \\ &= \frac{e^{jm\pi} - e^{-jm\pi}}{jm\omega_0} \\ &= \frac{2j\sin(m\pi)}{jm\omega_0} = \frac{2\sin(m\pi)}{m\omega_0} = 0, \qquad m \neq 0.\end{aligned}$$

The value is zero because the sine of a whole multiple of π is zero. For $m = 0$ the integrand is the constant 1, and the integral is the length of the period: $\int_{-T_0/2}^{T_0/2} 1\,\mathrm{d}t = T_0$.

ORTHOGONALITY OF COMPLEX EXPONENTIALS

$$\int_{T_0} e^{j(k-n)\omega_0 t}\,\mathrm{d}t = \begin{cases} T_0, & k = n \\ 0, & k \neq n \end{cases}$$

Only the term with $k = n$ survives in the sum, so $\int_{T_0} x(t)\, e^{-jn\omega_0 t}\,\mathrm{d}t = T_0\, a_n$. Divide both sides by T_0 and rename n as k. This is the analysis equation.

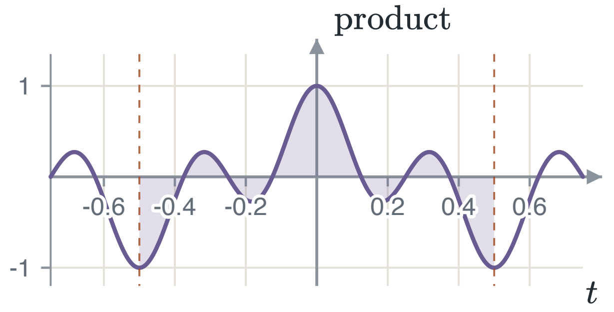

FIGURE 4.15 $\cos(2\pi t)\cos(4\pi t)$: two different harmonics of $T_0 = 1$. The lobes cancel and the area over one period is 0.

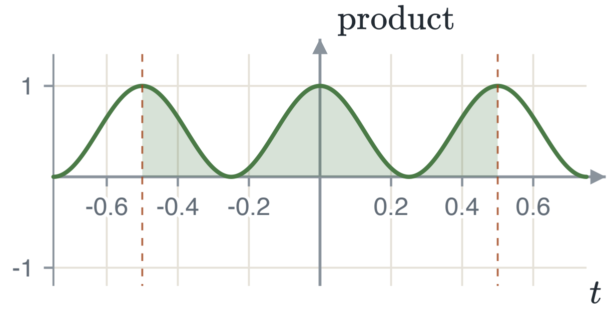

FIGURE 4.16 $\cos^2(2\pi t)$: a harmonic times itself never goes negative. The area over one period is $\frac{1}{2}$.

Two short consequences. If $x(t) = e^{j3\omega_0 t}$, the analysis integral is T_0 at $k = 3$ and zero otherwise, so $a_3 = 1$ and every other coefficient is zero. And with $T_0 = 1$, so $\omega_0 = 2\pi$, the integral $\int_0^1 e^{j2\pi t}e^{-j6\pi t}\,\mathrm{d}t$ has $m = 1 - 3 = -2 \neq 0$, so it is zero.

The DC coefficient

DC COEFFICIENT

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)\,\mathrm{d}t$$

Put $k = 0$ in the analysis equation, so that $e^0 = 1$. The coefficient a_0 is the average value of x over one period.

Find the average from the graph before any other coefficient. A formula for a_k must agree with it at $k = 0$. A wave that is 1 for half of each period and 0 for the other half has $a_0 = \frac{1}{2}$. A sawtooth that is odd about the middle of each period has $a_0 = 0$. For $x(t) = 3 + \cos 2t$ the period is $T_0 = 2\pi/2 = \pi$ s, and

$$a_0 = \frac{1}{\pi}\int_0^{\pi}\big(3 + \cos 2t\big)\,\mathrm{d}t = \frac{1}{\pi}\Big[3t + \tfrac{1}{2}\sin 2t\Big]_0^{\pi} = \frac{1}{\pi}\big(3\pi + 0 - 0\big) = 3.$$

⚠ ZERO FREQUENCY IS A HARMONIC

The $k = 0$ term is the constant a_0. Dropping it shifts the whole reconstruction by the average value.

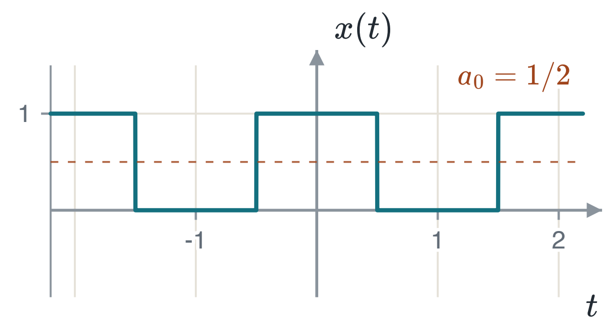

FIGURE 4.17 On for half of each period: $a_0 = \frac{1}{2}$.

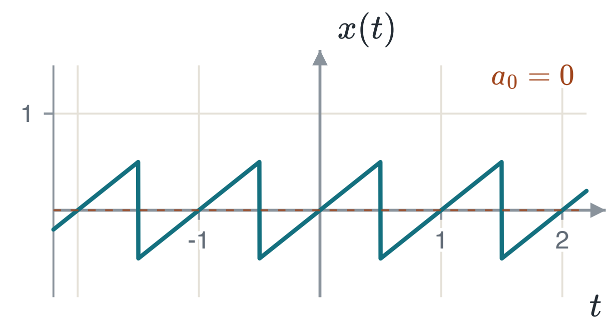

FIGURE 4.18 A sawtooth, odd about the middle of each period: $a_0 = 0$.

Real signals: cosine and sine forms

A real signal can be written with real functions only. The first step is a symmetry of the coefficients. For real x, conjugate the analysis equation. The conjugate of $x(t)$ is $x(t)$, and the conjugate of $e^{-jk\omega_0 t}$ is $e^{+jk\omega_0 t}$:

$$a_k^* = \frac{1}{T_0}\int_{T_0} x(t)\, e^{+jk\omega_0 t}\,\mathrm{d}t = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-j(-k)\omega_0 t}\,\mathrm{d}t = a_{-k}.$$

So $a_{-k} = a_k^*$. Now split the synthesis sum into $k = 0$, $k > 0$ and $k < 0$, and pair each k with $-k$. A number plus its conjugate is twice its real part, $z + z^* = 2\,\mathrm{Re}\{z\}$:

$$\begin{aligned} x(t) &= a_0 + \sum_{k=1}^{\infty}\left(a_k e^{jk\omega_0 t} + a_{-k}e^{-jk\omega_0 t}\right) \\ &= a_0 + \sum_{k=1}^{\infty}\Big(a_k e^{jk\omega_0 t} + \left(a_k e^{jk\omega_0 t}\right)^*\Big) \\ &= a_0 + 2\sum_{k=1}^{\infty}\mathrm{Re}\big\{a_k e^{jk\omega_0 t}\big\}. \end{aligned}$$

Two real forms follow, one for each way of writing a_k. In polar form, $a_k = A_k e^{j\theta_k}$, so $\mathrm{Re}\{A_k e^{j(k\omega_0 t+\theta_k)}\} = A_k\cos(k\omega_0 t + \theta_k)$. In rectangular form, $a_k = B_k + jC_k$. Put $\phi = k\omega_0 t$ and multiply out:

$$\begin{aligned} \mathrm{Re}\big\{(B_k + jC_k)(\cos\phi + j\sin\phi)\big\} &= \mathrm{Re}\big\{B_k\cos\phi - C_k\sin\phi + j(B_k\sin\phi + C_k\cos\phi)\big\} \\ &= B_k\cos\phi - C_k\sin\phi. \end{aligned}$$

REAL FORMS OF THE SERIES OF A REAL SIGNAL

$$a_k = A_k e^{j\theta_k}: \qquad x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k\cos\big(k\omega_0 t + \theta_k\big)$$

$$a_k = B_k + jC_k: \qquad x(t) = a_0 + 2\sum_{k=1}^{\infty}\big[B_k\cos k\omega_0 t - C_k\sin k\omega_0 t\big]$$

The first is the amplitude–phase form: each harmonic is one cosine with its own phase. The second is the cosine–sine form. The two are linked by $A_k = \sqrt{B_k^2 + C_k^2}$ and $\theta_k = \angle a_k$. Note the factor 2 and the minus sign before C_k.

EXAMPLE 4.6 — FROM COEFFICIENTS TO A REAL SIGNAL

GIVEN A real signal with $\omega_0 = \pi$ rad/s, $a_1 = 0.4 + j0.3$, $a_{-1} = a_1^*$ and every other $a_k = 0$. A second real signal with $a_1 = 1 - j$, $a_{-1} = 1 + j$ and every other $a_k = 0$.

FIND The first signal in both real forms, and the second in cosine–sine form.

METHOD Read B_1 and C_1 from the rectangular form of a_1. Compute A_1 and θ_1 from them. Substitute into the two real forms with $a_0 = 0$.

SOLUTION For the first signal, $B_1 = 0.4$ and $C_1 = 0.3$. Then $A_1 = \sqrt{0.4^2 + 0.3^2} = \sqrt{0.25} = 0.5$. The point $(0.4, 0.3)$ lies in the first quadrant, so $\theta_1 = \arctan(0.3/0.4) = 0.644$ rad. The two forms are

$$x(t) = 2A_1 \cos(\pi t + \theta_1) = \cos(\pi t + 0.644),$$
$$x(t) = 2\big[B_1 \cos \pi t - C_1 \sin \pi t\big] = 0.8 \cos \pi t - 0.6 \sin \pi t.$$

For the second signal, $B_1 = 1$ and $C_1 = -1$, so

$$x(t) = 2\big[1 \cdot \cos \pi t - (-1) \sin \pi t\big] = 2 \cos \pi t + 2 \sin \pi t.$$

CHECK Expand the amplitude–phase form with $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$. Here $\cos 0.644 = B_1/A_1 = 0.8$ and $\sin 0.644 = C_1/A_1 = 0.6$, so $\cos(\pi t + 0.644) = 0.8 \cos \pi t - 0.6 \sin \pi t$. The two forms agree.

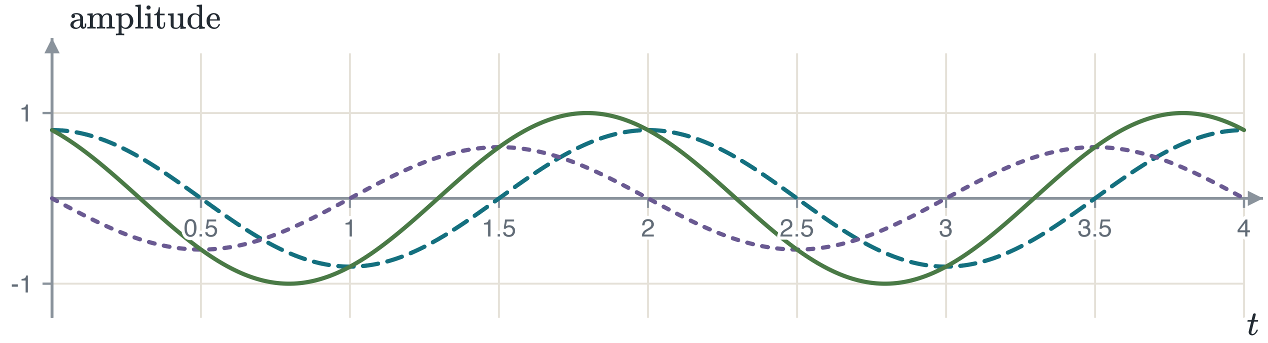

FIGURE 4.19 Example 4.6: $2B_1 \cos \pi t = 0.8 \cos \pi t$ (long dashes) and $-2C_1 \sin \pi t = -0.6 \sin \pi t$ (short dashes) add to one cosine of amplitude $2A_1 = 1$ (solid).

4.3 Series worked out

This section applies the analysis equation to three standard waveforms: the rectangular wave, the sawtooth and the impulse train. It also asks how many harmonics a good approximation needs.

EXAMPLE 4.7 – THE PERIODIC RECTANGULAR WAVE

GIVEN $x(t) = 1$ for $|t| < T_1$ and $x(t) = 0$ for $T_1 < |t| < T_0/2$, repeated with period T_0.

FIND Every coefficient.

METHOD Apply the analysis equation over $-T_0/2 < t < T_0/2$. The signal is 1 only on $-T_1 < t < T_1$, so those become the limits. Treat $k = 0$ on its own, because the antiderivative divides by k.

SOLUTION The signal is zero for $T_1 < |t| < T_0/2$, so those parts of the integral drop out and the limits shrink to $\pm T_1$:

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)\, e^{-jk\omega_0 t}\, \mathrm{d}t = \frac{1}{T_0} \int_{-T_1}^{T_1} 1 \cdot e^{-jk\omega_0 t}\, \mathrm{d}t.$$

For $k = 0$ the integrand is 1, so $a_0 = \frac{1}{T_0} \int_{-T_1}^{T_1} 1\, \mathrm{d}t = \frac{1}{T_0} \big[t\big]_{-T_1}^{T_1} = \frac{2T_1}{T_0}$. This is the fraction of each period that the pulse fills. For $k \neq 0$, integrate the exponential, evaluate it at the two limits, and turn the difference of exponentials into a sine with $e^{j\theta} - e^{-j\theta} = 2j \sin \theta$:

$$\begin{aligned} a_k &= \frac{1}{T_0}\left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right]_{-T_1}^{T_1} = \frac{1}{T_0}\cdot\frac{e^{-jk\omega_0 T_1} - e^{jk\omega_0 T_1}}{-jk\omega_0} \\ &= \frac{1}{T_0}\cdot\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{jk\omega_0} = \frac{1}{T_0}\cdot\frac{2j\sin(k\omega_0 T_1)}{jk\omega_0} \\ &= \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T_0}. \end{aligned}$$

The second line changes the sign of both numerator and denominator, which leaves the fraction unchanged. Finally replace ω_0 by $2\pi/T_0$. The denominator becomes $k\omega_0 T_0 = 2\pi k$ and the argument becomes $k\omega_0 T_1 = 2\pi k T_1/T_0$:

$$a_k = \begin{cases} \dfrac{2T_1}{T_0}, & k = 0, \\ \dfrac{\sin(2\pi k T_1/T_0)}{\pi k}, & k \neq 0. \end{cases}$$

CHECK

As $k \to 0$, $\sin\theta/\theta \to 1$, so the second branch tends to $\frac{2\pi k T_1/T_0}{\pi k} = 2T_1/T_0$. It agrees with the branch for $k = 0$. The signal is real and even, so the coefficients must be real and even in k. The formula has this property because $\sin$ and k are both odd.

⚠ HALF DUTY: NO EVEN HARMONICS

For $T_0 = 4T_1$ the pulse fills half of each period, and $a_k = \sin(\pi k/2)/(\pi k)$. At every even $k \neq 0$ the sine is the sine of a whole multiple of π, so **every even coefficient vanishes**. The first values are $a_0 = \frac{1}{2}$, $a_1 = 1/\pi = 0.318$, $a_2 = 0$ and $a_3 = \sin(3\pi/2)/(3\pi) = -1/(3\pi) = -0.106$.

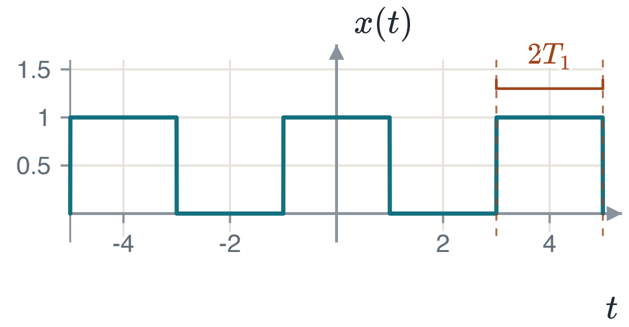

FIGURE 4.20 The wave for $T_1 = 1$ and $T_0 = 4$.

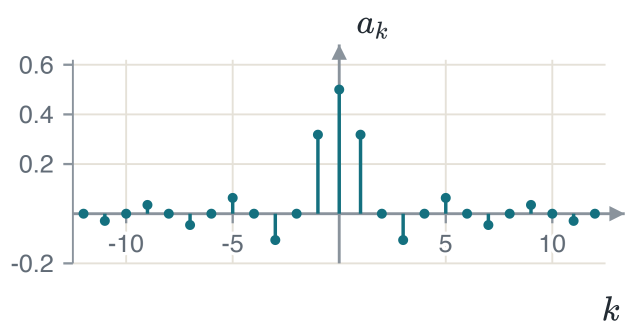

FIGURE 4.21 Its coefficients. They are real, so a negative stem is a phase of π.

The envelope, and the samples taken from it

These coefficients are values of one continuous function of ω, read at equally spaced points. Define the envelope by the integral of Example 4.7 with $k\omega_0$ replaced by a free variable ω:

ENVELOPE OF A SINGLE PULSE

$$E(\omega) = \int_{-T_1}^{T_1} e^{-j\omega t}\,\mathrm{d}t = \left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-T_1}^{T_1}$$
$$= \frac{e^{j\omega T_1} - e^{-j\omega T_1}}{j\omega} = \frac{2\sin(\omega T_1)}{\omega} = 2T_1\,\mathrm{sinc}(\omega T_1)$$

$$a_k = \frac{1}{T_0}\,E(k\omega_0) = \frac{1}{T_0}\cdot\frac{2\sin(k\omega_0 T_1)}{k\omega_0} = \frac{\sin(2\pi k T_1/T_0)}{\pi k}$$

Here $\mathrm{sinc}(\theta) = \sin\theta/\theta$, the unnormalised sinc. The second line puts $\omega = k\omega_0$ back and uses $\omega_0 = 2\pi/T_0$. E depends on the shape of one pulse only, not on how often the pulse repeats.

⊗ TWO OBJECTS, NOT ONE

$E(\omega)$ is a function of a continuous variable; a_k is a sequence indexed by an integer. Evaluating the envelope at the single point $\omega = \omega_0$ gives $T_0 a_1$, the first harmonic only. As a function of the harmonic index the coefficient is $T_0 a_k = T_0 \sin(2\pi k T_1/T_0)/(\pi k)$, with k in both places. Never join the tops of coefficient stems with a line: that claims the signal contains frequencies between the harmonics, which a periodic signal does not.

Keep the pulse and increase T_0. The envelope stays the same. The samples come closer together, because their spacing is $\omega_0 = 2\pi/T_0$, and each one scales by $1/T_0$. For example, $a_0 = 2T_1/T_0$ halves when T_0 doubles. Chapter 5 takes the limit $T_0 \to \infty$.

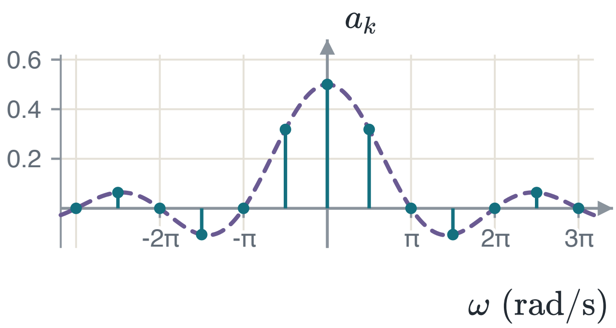

FIGURE 4.22 $T_0 = 4T_1$ with $T_1 = 1$: each a_k sits at $\omega = k\omega_0$ on the dashed envelope $E(\omega)/T_0$.

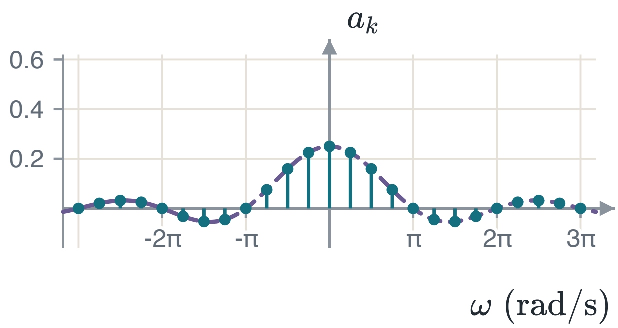

FIGURE 4.23 $T_0 = 8T_1$: the samples are twice as dense and half as tall.

How many harmonics are enough

A computer can add only finitely many terms. Keep the $2N+1$ coefficients nearest zero frequency and drop the rest. The mean-square error measures how far the result is from x: square the error, then average it over one period.

TRUNCATED SERIES AND MEAN-SQUARE ERROR

$$x_N(t) = \sum_{k=-N}^{N} a_k\, e^{jk\omega_0 t}, \qquad \mathrm{MSE} = \frac{1}{T_0}\int_{T_0} \left|x(t) - x_N(t)\right|^2 \mathrm{d}t$$

Section 4.5 shows that the MSE equals the power in the dropped harmonics, $\sum_{|k|>N} |a_k|^2$. The table uses that form.

TABLE 4.1 Mean-square error of the truncated series for the rectangular wave and the sawtooth.

N	RECTANGULAR WAVE, $T_0 = 4T_1$	SAWTOOTH, $T_0 = 1$
3	0.025	0.0144
9	0.010	0.0053
27	0.004	0.0018
81	0.001	0.0006

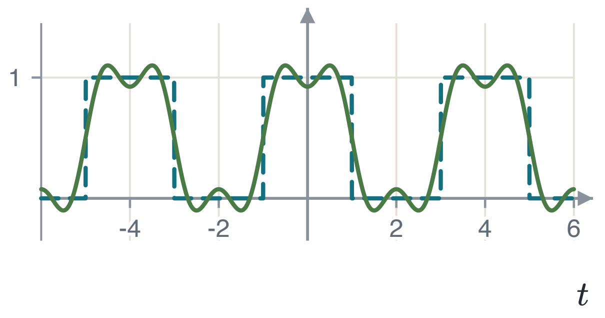

FIGURE 4.24 $N = 3$: the partial sum (solid) and the wave (dashed).

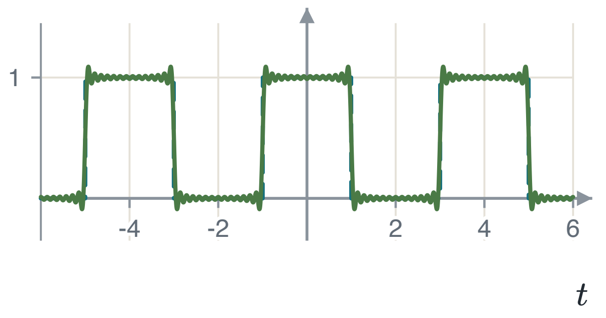

FIGURE 4.25 $N = 27$. The spike beside each jump is no shorter.

The rate at which the error falls is set by the smoothness of the signal. A jump makes the coefficients decay like $1/k$, so $|a_k|^2$ decays like $1/k^2$. A tail sum of this kind is close to the matching integral, $\sum_{k>N} 1/k^2 \approx \int_N^\infty \mathrm{d}k/k^2 = 1/N$. So the tail behaves like $1/N$, and doubling N roughly halves the error. A signal that is continuous but has a corner, such as a triangular wave, has coefficients decaying like $1/k^2$. Then $|a_k|^2$ decays like $1/k^4$, the tail like $\int_N^\infty \mathrm{d}k/k^4 = 1/(3N^3)$, and far fewer harmonics are needed.

⊗ THE GIBBS PHENOMENON

Near a jump the partial sum overshoots, and the overshoot does not shrink as N grows. Its height settles at about **9% of the size of the jump**. For the rectangular wave, which jumps by 1, the peak of x_N stays near 1.09. What shrinks is the width of the ripple: it moves into a narrower band around the jump. That is how the mean-square error can go to zero while the largest error does not. For a square wave that jumps from 0 to 2, the peak of x_{81} is about $2 + 0.09 \times 2 = 2.18$.

CONVERGENCE IN THE MEAN

If $x(t)$ has finite energy in one period, $\int_{T_0} |x(t)|^2\,\mathrm{d}t < \infty$, then the series converges in the mean: $\mathrm{MSE} \to 0$ as $N \to \infty$. This does not promise a small error at every t.

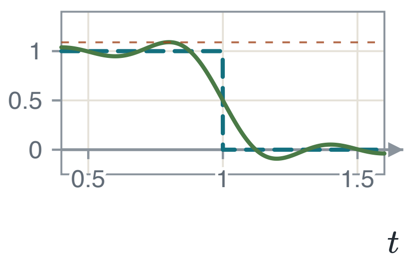

FIGURE 4.26 $N = 9$ near the jump at $t = 1$. The dashed line is at 1.09.

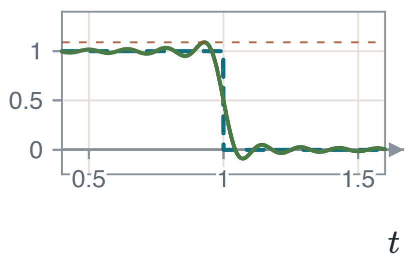

FIGURE 4.27 $N = 27$ near the jump at $t = 1$. The dashed line is at 1.09.

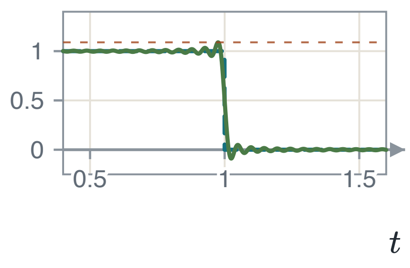

FIGURE 4.28 $N = 81$ near the jump at $t = 1$. The dashed line is at 1.09.

EXAMPLE 4.8 – THE SAWTOOTH WAVE

GIVEN $x(t) = t$ for $-T_0/2 < t < T_0/2$, repeated with period T_0.

FIND Every coefficient, with its magnitude and phase.

METHOD The DC term is a direct integral. For $k \neq 0$ the integrand is t times an exponential, so use integration by parts with $u = t$ and $\mathrm{d}v = e^{at}\,\mathrm{d}t$, which gives $\mathrm{d}u = \mathrm{d}t$ and $v = e^{at}/a$:

$$\int te^{at}\,\mathrm{d}t = \frac{te^{at}}{a} - \int \frac{e^{at}}{a}\,\mathrm{d}t = \frac{te^{at}}{a} - \frac{e^{at}}{a^2} = \frac{(at-1)e^{at}}{a^2}.$$

Differentiating the right side returns te^{at}, which checks the formula.

SOLUTION The DC term:

$$a_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} t\,\mathrm{d}t = \frac{1}{T_0}\left[\frac{t^2}{2}\right]_{-T_0/2}^{T_0/2} = \frac{1}{T_0}\left(\frac{T_0^2}{8} - \frac{T_0^2}{8}\right) = 0.$$

For $k \neq 0$, substitute the signal into the analysis equation and write $a = -jk\omega_0$, so that the by-parts formula applies:

$$a_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} t\,e^{-jk\omega_0 t}\,\mathrm{d}t = \frac{1}{T_0}\cdot\frac{1}{a^2}\Big[(at-1)e^{at}\Big]_{-T_0/2}^{T_0/2}.$$

At the limits, $aT_0/2 = -jk\omega_0 T_0/2 = -jk\pi$, because $\omega_0 T_0/2 = \pi$. Also $e^{\pm jk\pi} = \cos(k\pi) \pm j\sin(k\pi) = (-1)^k$, since $\sin(k\pi) = 0$. Both exponentials equal $(-1)^k$, and this common factor comes out of the bracket:

$$\begin{aligned}\Big[(at-1)e^{at}\Big]_{-T_0/2}^{T_0/2} &= (-jk\pi - 1)e^{-jk\pi} - (jk\pi - 1)e^{jk\pi} \\ &= (-1)^k\big[(-jk\pi - 1) - (jk\pi - 1)\big] \\ &= (-1)^k(-2jk\pi).\end{aligned}$$

Next, $a^2 = (-jk\omega_0)^2 = j^2k^2\omega_0^2 = -k^2\omega_0^2$. Divide, then use $\omega_0^2 = 4\pi^2/T_0^2$:

$$\begin{aligned} a_k &= \frac{1}{T_0}\cdot\frac{(-1)^k(-2jk\pi)}{-k^2\omega_0^2} = \frac{2j\pi(-1)^k}{T_0k\omega_0^2} \\ &= \frac{2j\pi(-1)^k\,T_0^2}{T_0k\cdot 4\pi^2} = \frac{jT_0(-1)^k}{2k\pi}, \qquad k \neq 0. \end{aligned}$$

The magnitude is $|a_k| = T_0/(2\pi|k|)$. Each a_k is purely imaginary, so its phase is $+\pi/2$ or $-\pi/2$.

CHECK The signal is real and odd, so the coefficients must be purely imaginary, and they are. Real gives $a_{-k} = a_k^*$ and odd gives $a_{-k} = -a_k$; the formula satisfies both, since replacing k by $-k$ changes only the sign. For $T_0 = 1$ the largest is $|a_{\pm1}| = 1/(2\pi) = 0.159$, and the magnitudes fall like $1/|k|$.

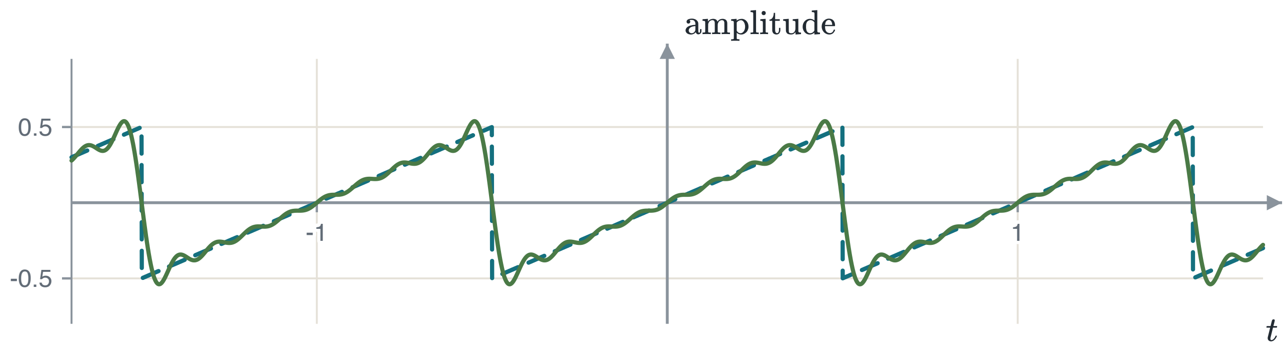

FIGURE 4.29 The sawtooth for $T_0 = 1$ (dashed) and its partial sum with $N = 9$ (solid). The same overshoot appears beside each jump.

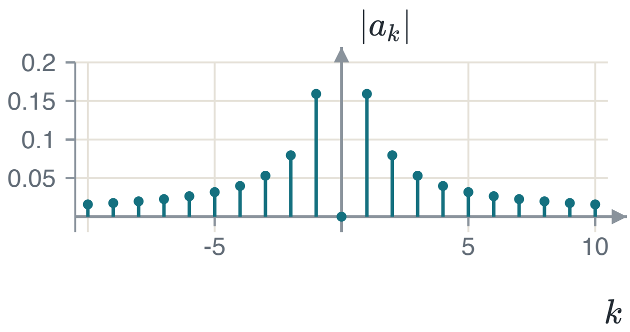

FIGURE 4.30 $|a_k| = 1/(2\pi|k|)$ for $T_0 = 1$.

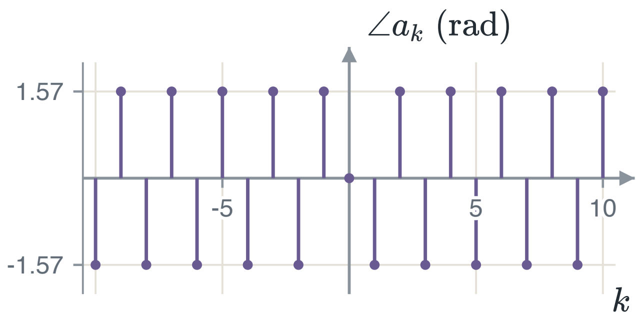

FIGURE 4.31 Every phase is $\pm\pi/2$, alternating in k.

EXAMPLE 4.9 — THE PERIODIC IMPULSE TRAIN

GIVEN $x(t) = \sum_{m=-\infty}^{\infty} \delta(t - mT_0)$.

FIND Every coefficient.

METHOD Take the period from $-T_0/2$ to $+T_0/2$. Exactly one impulse lies inside it, so the sifting property finishes the integral.

SOLUTION Substitute the impulse train into the analysis equation. Inside $-T_0/2 < t < T_0/2$ only the term $m = 0$, which is $\delta(t)$, is non-zero. The impulses at $t = \pm T_0, \pm 2T_0, \ldots$ lie outside the interval and contribute nothing. Then sift, which sets $t = 0$ in the exponential:

$$\begin{aligned} a_k &= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2}\sum_{m=-\infty}^{\infty}\delta(t - mT_0)\, e^{-jk\omega_0 t}\,\mathrm{d}t \\ &= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2}\delta(t)\, e^{-jk\omega_0 t}\,\mathrm{d}t = \frac{1}{T_0}\, e^{-jk\omega_0\cdot 0} = \frac{1}{T_0}\end{aligned}$$

for every k.

CHECK Both limits must be written and must differ. An interval with equal endpoints has zero length, so every integral over it is zero. The result is real, positive and the same at every harmonic: a flat spectrum with zero phase. At $t = 0$ all the harmonics equal 1 and add. At other times their contributions cancel. Chapter 7 uses this pair for sampling.

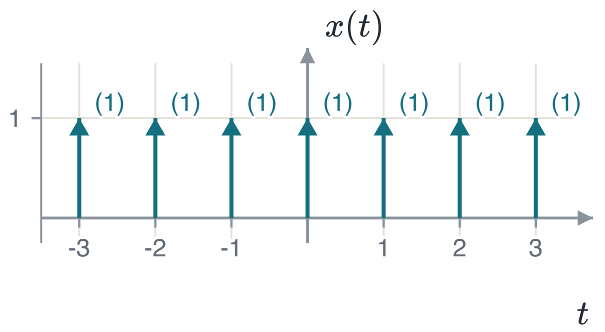

FIGURE 4.32 The impulse train for $T_0 = 1$. Each arrow is an impulse of weight 1.

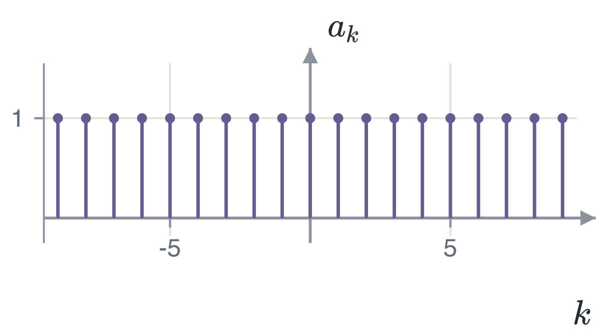

FIGURE 4.33 Its coefficients: $a_k = 1/T_0 = 1$ for every k.

4.4 The discrete-time series

A sequence is periodic with period N when $x[n] = x[n+N]$ for every n. Its fundamental frequency is $\omega_0 = 2\pi/N$ rad/sample. Its series has only N terms.

DISCRETE-TIME FOURIER SERIES

$$x[n] = \sum_{k=\langle N\rangle} a_k\, e^{jk(2\pi/N)n}$$

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\, e^{-jk(2\pi/N)n}$$

$\langle N\rangle$ means any N consecutive values of the index. Both sums are finite and there are exactly N distinct coefficients.

The analysis sum follows from a discrete orthogonality. Sum $e^{jm(2\pi/N)n}$ over one period, $n = 0, \ldots, N-1$, with m an integer. If m is a multiple of N, every term is $e^{j2\pi(m/N)n} = 1$ and the sum is N. Otherwise the terms form a geometric sum with ratio $r = e^{jm2\pi/N} \neq 1$, and $r^N = e^{jm2\pi} = 1$:

$$\sum_{n=0}^{N-1} r^n = \frac{1-r^N}{1-r} = \frac{1-e^{jm2\pi}}{1-r} = \frac{1-1}{1-r} = 0.$$

Now multiply the synthesis sum by $e^{-j\ell(2\pi/N)n}$ and sum over one period of n. Exchange the two finite sums. Within one period of k, the difference $k-\ell$ is a multiple of N only for $k=\ell$, so only that term survives:

$$\begin{aligned}\sum_{n=\langle N\rangle} x[n]\, e^{-j\ell(2\pi/N)n} &= \sum_{n=\langle N\rangle}\sum_{k=\langle N\rangle} a_k\, e^{j(k-\ell)(2\pi/N)n} \\ &= \sum_{k=\langle N\rangle} a_k \sum_{n=\langle N\rangle} e^{j(k-\ell)(2\pi/N)n} = N\, a_\ell.\end{aligned}$$

Divide by N and rename ℓ as k. This is the analysis sum.

> ⓘ THE COEFFICIENTS REPEAT
>
> $a_{k+N} = a_k$. Replace k by $k+N$ in the analysis sum and split the exponential:
>
> $$a_{k+N} = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\, e^{-jk(2\pi/N)n}\, e^{-j2\pi n} = a_k,$$
>
> because $e^{-j2\pi n} = 1$ for every integer n. For example, with $N=8$ and $a_3 = 0.2j$, $a_{11} = a_{3+8} = 0.2j$.

> ⊗ NOT IN CONTINUOUS TIME
>
> The same substitution in continuous time produces $e^{-j2\pi t}$, which is not 1 between integers. Continuous-time coefficients do not repeat.

> ✓ EXACT AND FINITE
>
> A period-N sequence is described by N numbers, and N terms rebuild it. There is no truncation, no limit, no question of convergence and no Gibbs phenomenon in discrete time.

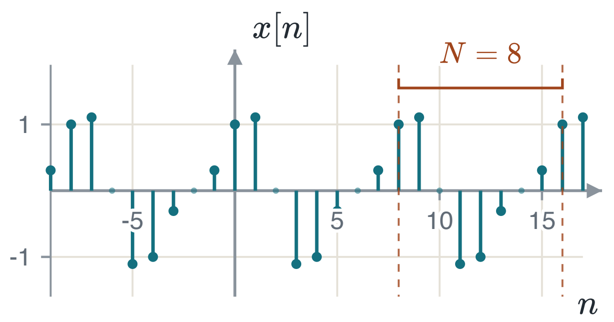

FIGURE 4.34 $x[n] = \cos(2\pi n/8) + 0.4\sin(4\pi n/8)$, with one period of $N=8$ marked.

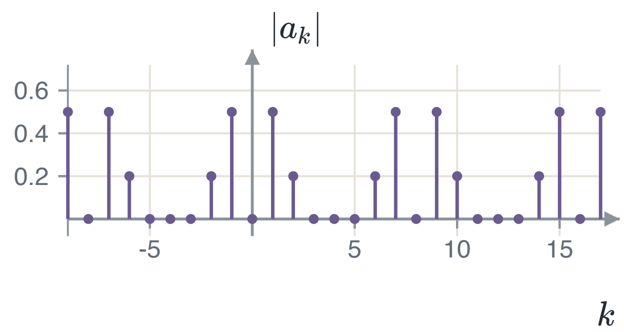

FIGURE 4.35 Its coefficient magnitudes, $\frac{1}{2}$ at $k=\pm1$ and 0.2 at $k=\pm2$, repeat every 8 in k.

The magnitudes in the figure come from Euler's relations. $\cos(2\pi n/8)$ gives $a_{\pm1} = \frac{1}{2}$. $0.4\sin(4\pi n/8) = 0.4\sin(2\cdot 2\pi n/8)$ gives $a_2 = \frac{0.4}{2j} = -0.2j$ and $a_{-2} = 0.2j$, both of magnitude

0.2.

EXAMPLE 4.10 – A SUM OF TWO SEQUENCES

GIVEN $x[n] = \sin\!\left(\frac{5\pi}{6}n\right) + \cos\!\left(\frac{3\pi}{4}n + \frac{\pi}{5}\right)$.

FIND The fundamental period and every coefficient.

METHOD A sinusoid of frequency ω repeats after $N = (2\pi/\omega)\,m$ samples, for the smallest positive integer m that makes N an integer. Find each component period, take the least common multiple, then expand with Euler's relations.

SOLUTION A sinusoid $\cos(\omega n)$ or $\sin(\omega n)$ repeats with period N when $\omega N = 2\pi m$ for some integer m, that is, $N = 2\pi m/\omega$. For $\omega = 5\pi/6$: $N = \frac{2\pi m}{5\pi/6} = \frac{12m}{5}$, first an integer at $m = 5$, giving $N = 12$. For $\omega = 3\pi/4$: $N = \frac{2\pi m}{3\pi/4} = \frac{8m}{3}$, first an integer at $m = 3$, giving $N = 8$. So $N_0 = \mathrm{LCM}(12, 8) = 24$ and $\omega_0 = 2\pi/24 = \pi/12$. Write each frequency as a multiple of ω_0:

$$\frac{5\pi}{6} = \frac{20\pi}{24} = 10\,\omega_0, \qquad \frac{3\pi}{4} = \frac{18\pi}{24} = 9\,\omega_0.$$

Now expand with Euler's relations. The phase $\pi/5$ stays with the coefficient, because $e^{j(9\omega_0 n+\pi/5)} = e^{j\pi/5}e^{j9\omega_0 n}$:

$$\sin(10\omega_0 n) = \tfrac{1}{2j}e^{j10\omega_0 n} - \tfrac{1}{2j}e^{-j10\omega_0 n},$$
$$\cos\!\left(9\omega_0 n + \tfrac{\pi}{5}\right) = \tfrac{1}{2}e^{j\pi/5}e^{j9\omega_0 n} + \tfrac{1}{2}e^{-j\pi/5}e^{-j9\omega_0 n}.$$

Read off the coefficient of each $e^{jk\omega_0 n}$:

$$a_{10} = \tfrac{1}{2j}, \quad a_{-10} = -\tfrac{1}{2j}, \quad a_9 = \tfrac{1}{2}e^{j\pi/5}, \quad a_{-9} = \tfrac{1}{2}e^{-j\pi/5}.$$

Every other coefficient in one period is zero, and $a_k = a_{k+24}$ gives all the rest.

CHECK $|a_{\pm 9}| = |a_{\pm 10}| = \frac{1}{2}$. The phases are $\angle a_{\pm 9} = \pm\pi/5$ and $\angle a_{\pm 10} = \mp\pi/2$, since $\frac{1}{2j} = -\frac{j}{2}$. The sequence is real, so the magnitudes are even in k and the phases odd.

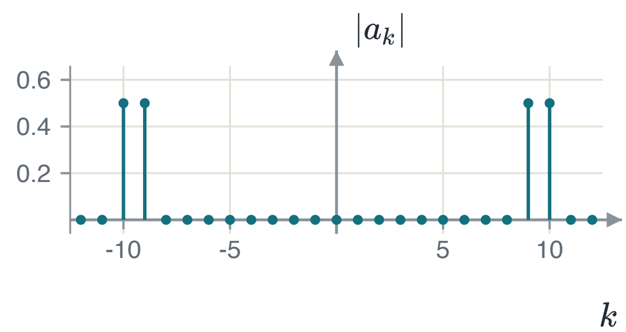

FIGURE 4.36 Magnitudes over one period, $-12 \le k \le 11$, with $k = 12$ equal to $k = -12$.

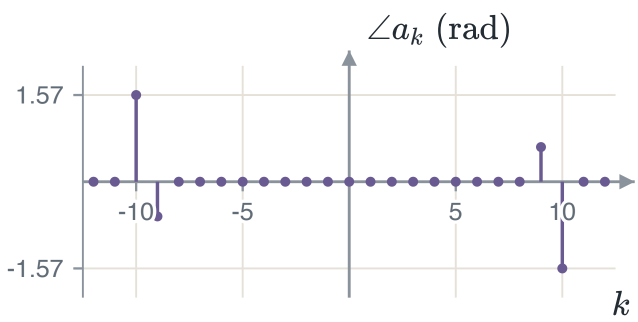

FIGURE 4.37 Phases: $\pm\pi/5$ at $k = \pm 9$, $\mp\pi/2$ at $k = \pm 10$.

EXAMPLE 4.11 – THE DISCRETE-TIME SQUARE WAVE

GIVEN $x[n] = 1$ for $|n| \leq N_1$ and $x[n] = 0$ for $N_1 < |n| \leq N/2$, repeated with period N.

FIND Every coefficient.

METHOD Sum over one period. The signal is 1 only for $-N_1 \leq n \leq N_1$, so the sum reduces to a finite geometric sum with ratio $r = e^{-jk(2\pi/N)}$. For $r \neq 1$ it has a closed form. Write $S = \sum_{n=m}^{p} r^n$. Then $S - rS = r^m - r^{p+1}$, because every other term cancels, so $S = (r^m - r^{p+1})/(1-r)$. Treat $r = 1$ separately.

SOLUTION With $m = -N_1$ and $p = N_1$,

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_1} 1 \cdot e^{-jk(2\pi/N)n} = \frac{1}{N}\sum_{n=-N_1}^{N_1} r^n = \frac{1}{N}\cdot\frac{r^{-N_1} - r^{N_1+1}}{1-r}.$$

The lower limit is $n = -N_1$, so the first numerator term is $r^{-N_1} = e^{+jk(2\pi/N)N_1}$, with a positive exponent. Written out,

$$a_k = \frac{1}{N}\cdot\frac{e^{jk(2\pi/N)N_1} - e^{-jk(2\pi/N)(N_1+1)}}{1 - e^{-jk(2\pi/N)}}.$$

Factor $e^{-jk\pi/N}$ out of the numerator. The exponents that remain are $jk(2\pi/N)N_1 + jk\pi/N = jk(2\pi/N)(N_1 + \frac{1}{2})$ and $-jk(2\pi/N)(N_1+1) + jk\pi/N = -jk(2\pi/N)(N_1 + \frac{1}{2})$, so the bracket is $2j$ times a sine:

$$\begin{aligned} &e^{jk(2\pi/N)N_1} - e^{-jk(2\pi/N)(N_1+1)} \\ &\quad = e^{-jk\pi/N}\left[e^{jk(2\pi/N)(N_1+\frac{1}{2})} - e^{-jk(2\pi/N)(N_1+\frac{1}{2})}\right] \\ &\quad = e^{-jk\pi/N}\, 2j\sin\!\left(\tfrac{2\pi k}{N}\left(N_1 + \tfrac{1}{2}\right)\right). \end{aligned}$$

Factor the same $e^{-jk\pi/N}$ out of the denominator:

$$1 - e^{-jk(2\pi/N)} = e^{-jk\pi/N}\left[e^{jk\pi/N} - e^{-jk\pi/N}\right] = e^{-jk\pi/N}\, 2j\sin\!\left(\tfrac{\pi k}{N}\right).$$

The factors $e^{-jk\pi/N}$ and $2j$ cancel. When k is a multiple of N, $r = e^{-j2\pi(k/N)} = 1$, every term of the sum is 1, and there are $2N_1 + 1$ terms. So

$$a_k = \begin{cases} \dfrac{2N_1+1}{N}, & k = 0, \pm N, \pm 2N, \ldots \\[2ex] \dfrac{\sin\!\left(\frac{2\pi k}{N}(N_1 + \frac{1}{2})\right)}{N\sin(\pi k/N)}, & \text{otherwise.} \end{cases}$$

CHECK Here $|r| = 1$ at every k: r lies on the unit circle. So the rule $|r| < 1$ of an infinite geometric sum does not apply, and it is not needed, because a finite sum needs only $r \neq 1$. For $N = 10$ and $N_1 = 2$ the first branch gives $a_0 = (2\cdot 2 + 1)/10 = 0.5$, the average of five ones in ten samples. As $k \to 0$ the second branch tends to $\frac{(2\pi k/N)(N_1+\frac{1}{2})}{N\pi k/N} = \frac{2N_1+1}{N}$, the same value.

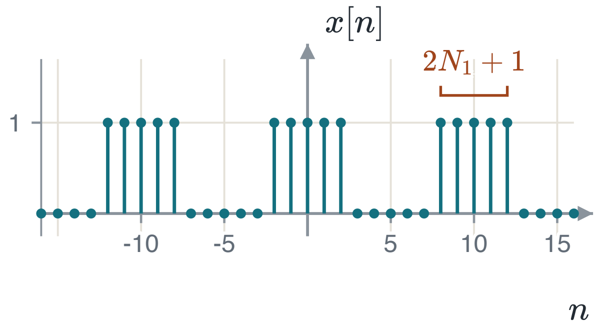

FIGURE 4.38 $N = 10$, $N_1 = 2$: five ones and five zeros in every period.

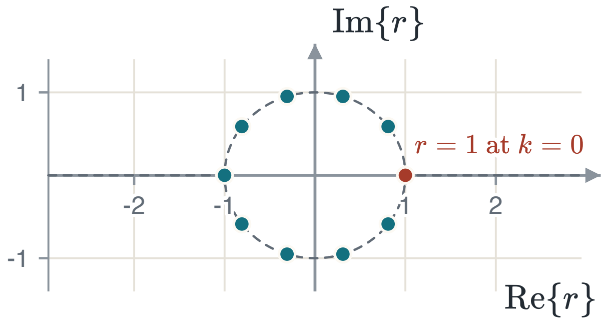

FIGURE 4.39 The ten values of $r = e^{-jk2\pi/10}$. All have $|r| = 1$; only $k = 0$ gives $r = 1$.

Each parameter has one job. The $2N_1 + 1$ ones set the height and the width of each lobe. The period N sets how far apart the peaks repeat in k, and it scales every coefficient by $1/N$. With $N_1 = 2$ the peak is $5/N$: 0.5 for $N = 10$, 0.25 for $N = 20$ and 0.167 for $N = 30$. With $N = 12$ and $N_1 = 1$, $a_0 = (2 \cdot 1 + 1)/12 = \frac{3}{12} = \frac{1}{4}$.

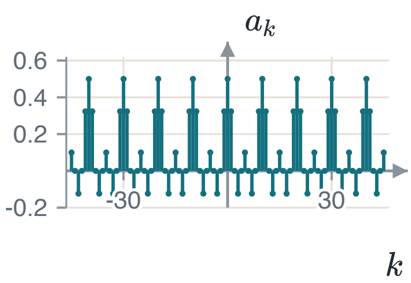

FIGURE 4.40 $N = 10$, $N_1 = 2$.

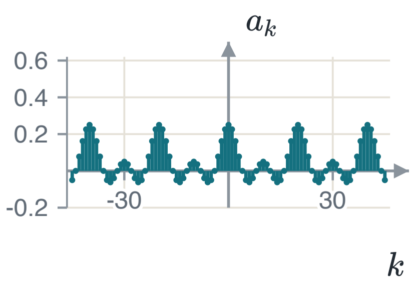

FIGURE 4.41 $N = 20$, $N_1 = 2$.

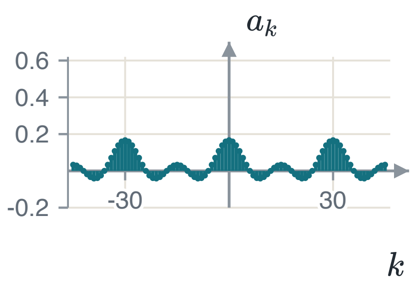

FIGURE 4.42 $N = 30$, $N_1 = 2$.

Rebuild $x[n]$ from all N coefficients with the synthesis sum. Every sample lands on 1 or 0 exactly. The sum has N terms and stops. Nothing is truncated, so there is no overshoot that refuses to shrink.

EXAMPLE 4.12 — A DISCRETE SAWTOOTH

GIVEN $x[n] = n$ for $-5 \le n \le 5$, repeated with period $N = 11$.

FIND Every coefficient.

METHOD Substitute into the analysis sum with $\omega_0 = 2\pi/11$. The sequence is odd, so pair the term at $n = m$ with the term at $n = -m$ before expanding. With $e^{j\theta} - e^{-j\theta} = 2j\sin\theta$,

$$m\,e^{-jk\omega_0 m} + (-m)\,e^{+jk\omega_0 m} = -m\big(e^{jk\omega_0 m} - e^{-jk\omega_0 m}\big) = -2jm\sin(k\omega_0 m).$$

SOLUTION The DC term is the average of the eleven values, and they cancel in pairs:

$$a_0 = \frac{1}{11}\sum_{n=-5}^{5} n = \frac{1}{11}\big[(-5) + (-4) + \cdots + 4 + 5\big] = 0.$$

For general k, the $n = 0$ term is zero, and the remaining ten terms form five pairs:

$$
\begin{aligned}
a_k &= \frac{1}{11}\sum_{n=-5}^{5} n\, e^{-jk(2\pi/11)n} \\
&= \frac{1}{11}\sum_{m=1}^{5}\Big[m\, e^{-jk(2\pi/11)m} + (-m)\, e^{+jk(2\pi/11)m}\Big] \\
&= \frac{1}{11}\sum_{m=1}^{5}(-2jm)\sin\left(\frac{2\pi k m}{11}\right) \\
&= -\frac{2j}{11}\sum_{m=1}^{5} m \sin\left(\frac{2\pi k m}{11}\right).
\end{aligned}
$$

For $k = 1$ the five terms are $1\sin\frac{2\pi}{11} + 2\sin\frac{4\pi}{11} + 3\sin\frac{6\pi}{11} + 4\sin\frac{8\pi}{11} + 5\sin\frac{10\pi}{11} = 0.5406 + 1.8193 + 2.9695 + 3.0230 + 1.4087 = 9.7610$, so $a_1 = -j\,\frac{2}{11}(9.7610) = -j1.7747$.

CHECK Every coefficient is purely imaginary, as a real odd sequence requires, so each phase is $+\pi/2$ or $-\pi/2$. The sequence is odd, so $a_{-1} = -a_1 = +j1.7747$. The largest magnitude is $|a_{\pm 1}| = 1.7747$, and it returns at $k = \pm 10, \pm 12, \ldots$ because $a_{k+11} = a_k$. The eleven coefficients from $k = -5$ to $k = 5$ are the whole answer.

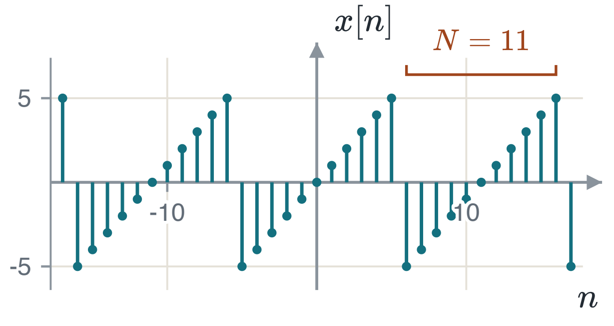

FIGURE 4.43 A ramp from -5 to 5, repeated every 11 samples.

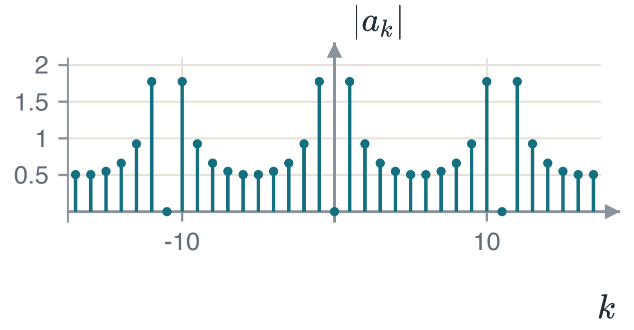

FIGURE 4.44 $|a_k|$: largest at $k = \pm 1$, repeating every 11.

4.5 Properties

Each property states how an operation on a periodic signal changes its coefficients. The notation $x \leftrightarrow a_k$ pairs a periodic signal with its coefficients. Throughout, x and y have the same period, T_0 or N, with coefficients a_k and b_k. Section 4.7 collects every property in two tables.

Linearity and time shift

The analysis equation is an integral, and an integral is linear, so $Ax(t) + By(t) \leftrightarrow Aa_k + Bb_k$. The same holds for sequences. For a shift, substitute $\tau = t - t_0$, so $t = \tau + t_0$ and $\mathrm{d}t = \mathrm{d}\tau$. When t runs over the period $0 < t < T_0$, τ runs over $-t_0 < \tau < T_0 - t_0$, which is again one full period. The factor $e^{-jk\omega_0 t_0}$ does not depend on τ and leaves the integral:

$$\frac{1}{T_0}\int_0^{T_0} x(t-t_0)\,e^{-jk\omega_0 t}\,\mathrm{d}t = \frac{1}{T_0}\int_{-t_0}^{T_0-t_0} x(\tau)\,e^{-jk\omega_0(\tau+t_0)}\,\mathrm{d}\tau$$
$$= e^{-jk\omega_0 t_0}\,\frac{1}{T_0}\int_{-t_0}^{T_0-t_0} x(\tau)\,e^{-jk\omega_0\tau}\,\mathrm{d}\tau = e^{-jk\omega_0 t_0}\,a_k.$$

TIME SHIFT

$$x(t-t_0) \;\longleftrightarrow\; a_k\,e^{-jk\omega_0 t_0}, \qquad x[n-n_0] \;\longleftrightarrow\; a_k\,e^{-jk(2\pi/N)n_0}$$

The discrete-time proof is the same, with a sum and the index change $r = n - n_0$.

The factor has modulus 1, $\left|e^{-jk\omega_0 t_0}\right| = 1$. So a shift never changes a magnitude, $|b_k| = |a_k|$ at every k. It changes each phase by $-k\omega_0 t_0$, in proportion to the harmonic index. Two signals that differ only by a delay have the same magnitude plot. Everything that tells them apart is in the phase. For a wave with $T_0 = 2$ ($\omega_0 = \pi$ rad/s) delayed by $t_0 = 0.4$, the phase of a_1 changes by $-1\cdot\pi\cdot 0.4 = -0.4\pi$. A delay of half a period, $t_0 = T_0/2$, multiplies a_1 by $e^{-j\omega_0 T_0/2} = e^{-j\pi} = -1$.

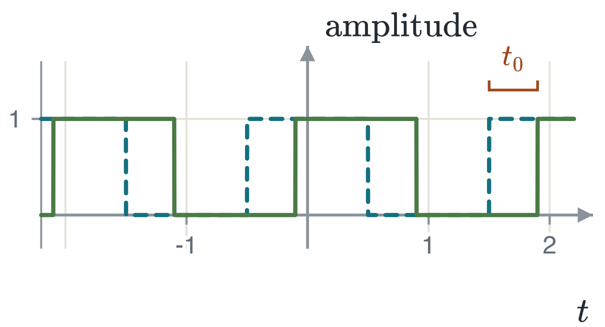

FIGURE 4.45 A rectangular wave with $T_0 = 2$ (dashed), delayed by $t_0 = 0.4$ (solid).

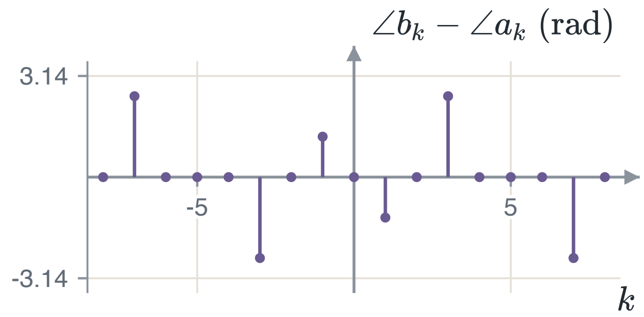

FIGURE 4.46 The phase change $-0.4\pi k$, drawn in $[-\pi,\pi]$. Where $a_k = 0$ there is no phase to change.

Time reversal and conjugation

For time reversal, take the period from $-T_0/2$ to $T_0/2$ and substitute $\tau = -t$, so $\mathrm{d}t = -\mathrm{d}\tau$. The limit $t = -T_0/2$ becomes $\tau = T_0/2$, and $t = T_0/2$ becomes $\tau = -T_0/2$. Swapping the limits removes the minus sign:

$$\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(-t)\,e^{-jk\omega_0 t}\,\mathrm{d}t = \frac{1}{T_0}\int_{T_0/2}^{-T_0/2} x(\tau)\,e^{jk\omega_0\tau}\,(-\mathrm{d}\tau)$$
$$= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(\tau)\,e^{-j(-k)\omega_0\tau}\,\mathrm{d}\tau = a_{-k}.$$

For conjugation, conjugate the synthesis equation term by term, then rename the index k as $-k$. The sum runs over all integers, so renaming does not change it:

$$x^*(t) = \sum_{k=-\infty}^{\infty} a_k^*\,e^{-jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_{-k}^*\,e^{jk\omega_0 t}.$$

REVERSAL AND CONJUGATION

$$x(-t) \longleftrightarrow a_{-k}, \qquad x^*(t) \longleftrightarrow a^*_{-k}$$

The same two rules hold for $x[-n]$ and $x^*[n]$.

These two rules give the symmetry of the coefficients. The coefficients of a signal are unique, because the analysis equation gives one value for each k. So a signal equal to its own conjugate, or to its own reversal, must have matching coefficients.

- **Real:** $x = x^*$, so $a_k = a^*_{-k}$, that is, $a_{-k} = a^*_k$. Then $|a_k|$ is even in k and $\angle a_k$ is odd.
- **Real and even:** even gives $a_{-k} = a_k$. With $a_{-k} = a^*_k$ this is $a_k = a^*_k$, so every a_k is real, and even in k.
- **Real and odd:** odd, $x(-t) = -x(t)$, gives $a_{-k} = -a_k$. With $a_{-k} = a^*_k$ this is $a^*_k = -a_k$, so $\mathrm{Re}\{a_k\} = 0$: every a_k is purely imaginary, and odd in k.

The rectangular wave of Example 4.7 is real and even, and its coefficients are real. The sawtooth of Example 4.8 is real and odd, and its coefficients are purely imaginary. A real, even signal cannot have $a_2 = 0.1 + 0.2j$, because that number is not real.

A real signal that is neither even nor odd splits into an even part and an odd part. Write each part as a combination of $x(t)$ and $x(-t)$. Apply linearity and time reversal, and then use $a_{-k} = a^*_k$ for a real signal:

$$\begin{aligned}
\mathcal{E}\mathrm{v}\{x(t)\} &= \tfrac{1}{2}x(t) + \tfrac{1}{2}x(-t) \;\leftrightarrow\; \tfrac{1}{2}a_k + \tfrac{1}{2}a_{-k} = \tfrac{1}{2}\big(a_k + a^*_k\big) = \mathrm{Re}\{a_k\}, \\
\mathcal{O}\mathrm{dd}\{x(t)\} &= \tfrac{1}{2}x(t) - \tfrac{1}{2}x(-t) \;\leftrightarrow\; \tfrac{1}{2}a_k - \tfrac{1}{2}a_{-k} = \tfrac{1}{2}\big(a_k - a^*_k\big) = j\,\mathrm{Im}\{a_k\}.
\end{aligned}$$

The last equality on each line uses $a_k + a^*_k = 2\,\mathrm{Re}\{a_k\}$ and $a_k - a^*_k = 2j\,\mathrm{Im}\{a_k\}$. So the even part of a real signal carries the real parts of the coefficients, and the odd part carries the imaginary parts. For example, if x is real and $a_1 = 0.3 - 0.4j$, the coefficient of $\mathcal{E}\mathrm{v}\{x\}$ at $k = 1$ is 0.3 and the coefficient of $\mathcal{O}\mathrm{dd}\{x\}$ is $j(-0.4) = -0.4j$. For a real signal the coefficients at negative k carry nothing new: compute them for $k \geq 0$ and conjugate.

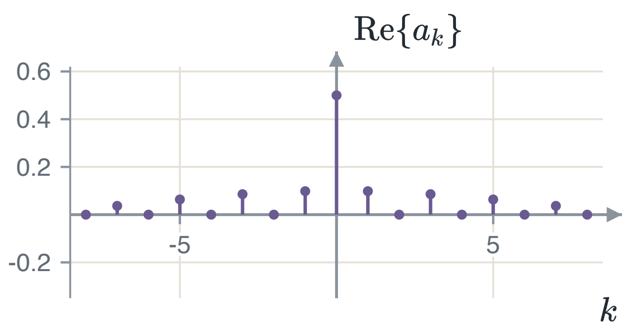

FIGURE 4.47 The delayed wave of the previous figure: $\mathrm{Re}\{a_k\}$, even in k.

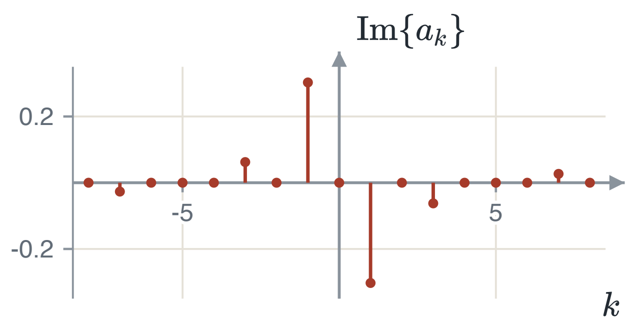

FIGURE 4.48 $\mathrm{Im}\{a_k\}$, odd in k.

Frequency shift and time scaling

Multiply $x(t)$ by the harmonic $e^{jM\omega_0 t}$, with M an integer, so the product keeps the period T_0. Rename the index as $k = \ell + M$:

$$e^{jM\omega_0 t}\sum_{\ell=-\infty}^{\infty} a_\ell\, e^{j\ell\omega_0 t} = \sum_{\ell=-\infty}^{\infty} a_\ell\, e^{j(\ell+M)\omega_0 t} = \sum_{k=-\infty}^{\infty} a_{k-M}\, e^{jk\omega_0 t}.$$

So $e^{jM\omega_0 t}x(t) \leftrightarrow a_{k-M}$: the whole coefficient sequence slides M places. In discrete time, $e^{jM(2\pi/N)n}x[n] \leftrightarrow a_{k-M}$ by the same steps.

Time scaling in continuous time, $x(\alpha t)$ with $\alpha > 0$, changes the period to T_0/α and the fundamental frequency to $\alpha\omega_0$. Substitute αt into the synthesis equation:

$$x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0(\alpha t)} = \sum_{k=-\infty}^{\infty} a_k\, e^{jk(\alpha\omega_0)t}.$$

This is a Fourier series with fundamental frequency $\alpha\omega_0$ and the same coefficients. The coefficients are unique, so they are the coefficients of $x(\alpha t)$. Each a_k keeps its value and moves to the frequency $k\alpha\omega_0$. For example, if $x(t)$ has $a_3 = 0.3$, then $x(2t)$ also has 0.3 at $k = 3$, now at the frequency $6\omega_0$.

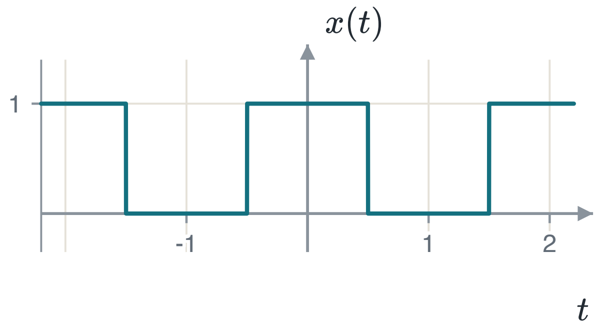

FIGURE 4.49 $x(t)$ with $T_0 = 2$.

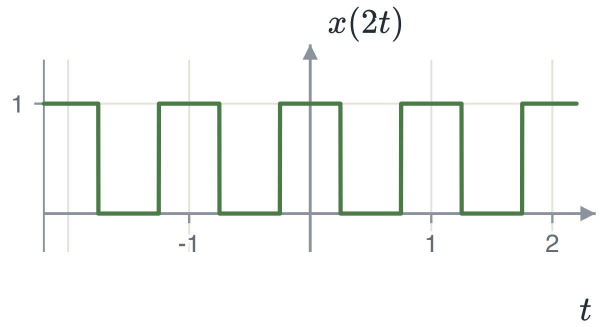

FIGURE 4.50 $x(2t)$: the period halves to 1 and ω_0 doubles.

Discrete time has no $x(\alpha n)$ for a fractional α. Its version of stretching inserts zeros. For a positive integer m, define

$$x_{(m)}[n] = \begin{cases} x[n/m], & n \text{ a multiple of } m, \\ 0, & \text{otherwise.} \end{cases}$$

Every sample of x is kept and $m-1$ zeros follow each one, so the period becomes mN. In the analysis sum over one period of mN samples, only $n = mr$ with $r = 0, \ldots, N-1$ contributes, and there $x_{(m)}[mr] = x[r]$. The exponent simplifies because $\frac{2\pi}{mN}\cdot mr = \frac{2\pi}{N}r$:

$$\begin{aligned} b_k &= \frac{1}{mN}\sum_{n=0}^{mN-1} x_{(m)}[n]\, e^{-jk\frac{2\pi}{mN}n} = \frac{1}{mN}\sum_{r=0}^{N-1} x[r]\, e^{-jk\frac{2\pi}{mN}mr} \\ &= \frac{1}{m}\cdot\frac{1}{N}\sum_{r=0}^{N-1} x[r]\, e^{-jk\frac{2\pi}{N}r} = \frac{a_k}{m}. \end{aligned}$$

⚠ NOT LIKE CONTINUOUS TIME

$x(\alpha t)$ keeps its coefficients. $x_{(m)}[n]$ divides them by m, because the same samples are averaged over m times as many points. For example, $x[n] = 1, 0$ repeating has $N = 2$ and $a_0 = \frac{1}{2}$. Stretched with $m = 3$ it becomes $1, 0, 0, 0, 0, 0$ repeating, with $a_0 = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$: one 1 in six samples.

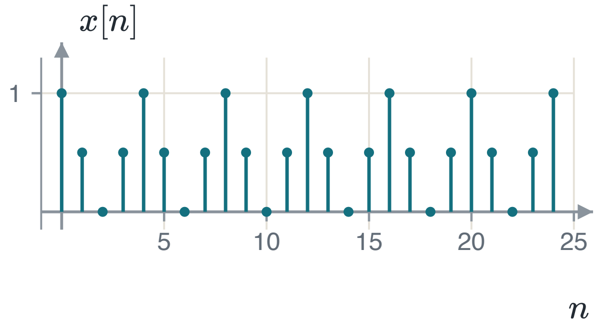

FIGURE 4.51 $x[n]$ with $N = 4$.

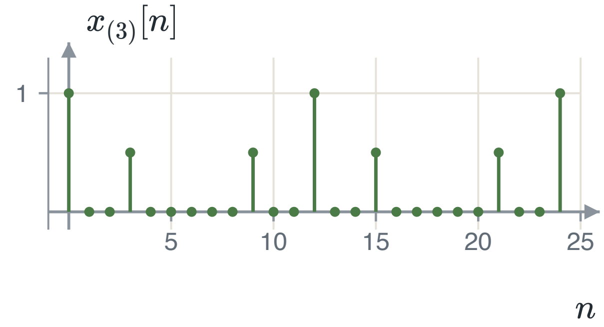

FIGURE 4.52 $x_{(3)}[n]$: two zeros after each sample, period 12.

Periodic convolution

Two periodic signals cannot be convolved over all time: the integral would not converge. Integrate over one period instead. The result $z(t) = \int_{T_0} x(\tau)\, y(t-\tau)\, \mathrm{d}\tau$ has period T_0. To find its coefficients, substitute z into the analysis equation and exchange the two integrals. In the inner integral put $\sigma = t - \tau$ with τ fixed; t over one period gives σ over one period, and $e^{-jk\omega_0 t} = e^{-jk\omega_0\sigma} e^{-jk\omega_0\tau}$:

$$\begin{aligned} c_k &= \frac{1}{T_0} \int_{T_0} \Big[\int_{T_0} x(\tau)\, y(t-\tau)\, \mathrm{d}\tau \Big] e^{-jk\omega_0 t}\, \mathrm{d}t \\ &= \frac{1}{T_0} \int_{T_0} x(\tau) \Big[\int_{T_0} y(t-\tau)\, e^{-jk\omega_0 t}\, \mathrm{d}t \Big] \mathrm{d}\tau \\ &= \frac{1}{T_0} \int_{T_0} x(\tau)\, e^{-jk\omega_0\tau} \Big[\int_{T_0} y(\sigma)\, e^{-jk\omega_0\sigma}\, \mathrm{d}\sigma \Big] \mathrm{d}\tau \\ &= \frac{1}{T_0} \int_{T_0} x(\tau)\, e^{-jk\omega_0\tau} \left[T_0\, b_k \right] \mathrm{d}\tau = T_0\, b_k\, a_k. \end{aligned}$$

PERIODIC CONVOLUTION

$$\int_{T_0} x(\tau)\, y(t-\tau)\, \mathrm{d}\tau \;\longleftrightarrow\; T_0\, a_k\, b_k, \qquad \sum_{r=\langle N \rangle} x[r]\, y[n-r] \;\longleftrightarrow\; N\, a_k\, b_k$$

The discrete-time proof is the same with sums, and the factor is N. Keep the factor T_0 or N: without it the shape of the result is right and its size is wrong.

✎ EXAMPLE 4.13 — A RECTANGULAR WAVE CONVOLVED WITH ITSELF

GIVEN The rectangular wave with $T_0 = 2$ and $T_1 = 0.5$: $x(t) = 1$ for $|t| < 0.5$ and 0 for $0.5 < |t| < 1$, repeated. Let $z(t) = \int_{T_0} x(\tau)\, x(t-\tau)\, \mathrm{d}\tau$.

FIND The coefficients of z, its mean, and its shape.

METHOD Here $T_0 = 4T_1$, so Example 4.7 gives $a_0 = \frac{1}{2}$ and $a_k = \sin(\pi k/2)/(\pi k)$. Apply the periodic-convolution property with $b_k = a_k$. For the shape, compute the overlap of the two pulses directly.

SOLUTION The property gives $c_k = T_0 a_k^2 = 2a_k^2$. So $c_0 = 2 \cdot \frac{1}{4} = \frac{1}{2}$. For odd k, $\sin^2(\pi k/2) = 1$ and $c_k = 2/(\pi^2 k^2)$. For even $k \neq 0$, $c_k = 0$. The mean of z is $c_0 = \frac{1}{2}$. For the shape, take τ over the period $-1 < \tau < 1$ and $0 \le t \le 1$. The first pulse is 1 on $-\frac{1}{2} < \tau < \frac{1}{2}$. The second, $x(t-\tau)$, is 1 on $t - \frac{1}{2} < \tau < t + \frac{1}{2}$. They overlap on $t - \frac{1}{2} < \tau < \frac{1}{2}$, of length $1 - t$. By symmetry the overlap is $1 + t$ for $-1 \le t \le 0$. So $z(t) = 1 - |t|$ on $|t| \le 1$, repeated every 2: a triangular wave with peak 1.

CHECK The mean of the triangle over one period is $\frac{1}{2}\int_{-1}^{1}(1-|t|)\,\mathrm{d}t = \frac{1}{2}\cdot 2\left[t - \frac{t^2}{2}\right]_0^1 = \frac{1}{2}\cdot 2 \cdot \frac{1}{2} = \frac{1}{2} = c_0$. Also compute c_1 of the triangle directly. The triangle is even, so $c_1 = \frac{1}{2}\int_{-1}^{1}(1-|t|)\cos(\pi t)\,\mathrm{d}t = \int_0^1 (1-t)\cos(\pi t)\,\mathrm{d}t$. By parts with $u = 1 - t$ and $\mathrm{d}v = \cos(\pi t)\,\mathrm{d}t$: $\left[(1-t)\frac{\sin \pi t}{\pi}\right]_0^1 + \int_0^1 \frac{\sin \pi t}{\pi}\,\mathrm{d}t = 0 + \frac{1}{\pi}\left[-\frac{\cos \pi t}{\pi}\right]_0^1 = \frac{2}{\pi^2}$. The property gives $2a_1^2 = 2/\pi^2$.

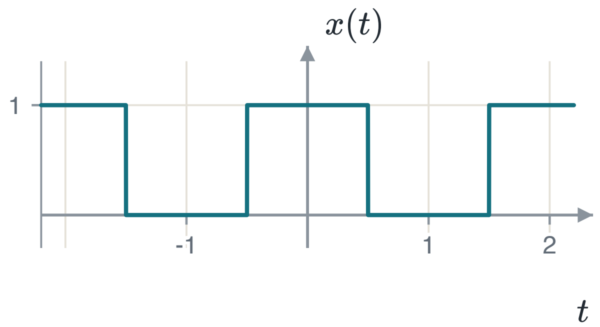

FIGURE 4.53 The rectangular wave, $T_0 = 2$, $T_1 = 0.5$.

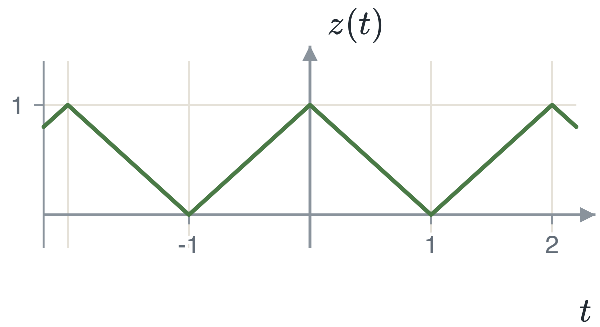

FIGURE 4.54 Its periodic convolution with itself: a triangular wave, peak 1, mean $\frac{1}{2}$.

Multiplication

Multiply the two synthesis sums. Use a second index m for the second sum, so that the two stay separate. Then collect the terms with the same total index $k = \ell + m$, that is, $m = k - \ell$:

$$x(t)\,y(t) = \sum_{\ell=-\infty}^{\infty} a_\ell\, e^{j\ell\omega_0 t} \sum_{m=-\infty}^{\infty} b_m\, e^{jm\omega_0 t} = \sum_{\ell}\sum_{m} a_\ell\, b_m\, e^{j(\ell+m)\omega_0 t}$$
$$= \sum_{k=-\infty}^{\infty}\left(\sum_{\ell=-\infty}^{\infty} a_\ell\, b_{k-\ell}\right) e^{jk\omega_0 t}.$$

The indices ℓ and $k - \ell$ add to k, so the new coefficients are a convolution of the two coefficient sequences. In discrete time both synthesis sums run over one period. For fixed ℓ, as m runs over one period so does $k = \ell + m$, and b repeats every N, so

$$x[n]\,y[n] = \sum_{\ell=\langle N\rangle}\sum_{m=\langle N\rangle} a_\ell\, b_m\, e^{j(\ell+m)(2\pi/N)n} = \sum_{k=\langle N\rangle}\left(\sum_{\ell=\langle N\rangle} a_\ell\, b_{k-\ell}\right) e^{jk(2\pi/N)n}.$$

MULTIPLICATION

$$x(t)\,y(t) \;\longleftrightarrow\; \sum_{\ell=-\infty}^{\infty} a_\ell\, b_{k-\ell}, \qquad x[n]\,y[n] \;\longleftrightarrow\; \sum_{\ell=\langle N\rangle} a_\ell\, b_{k-\ell}$$

ⓧ ONE PERIOD OF ℓ

The discrete-time product sums over one period of ℓ only: it is a **periodic** convolution of the coefficients. Discrete-time coefficients repeat, so a sum over all ℓ would add the same N products again and again and diverge.

For example, take $x(t) = y(t) = \cos(\omega_0 t)$, with $a_{\pm 1} = \frac{1}{2}$ and every other $a_k = 0$. The product $\cos^2(\omega_0 t)$ has $c_0 = a_1 a_{-1} + a_{-1} a_1 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ and $c_{\pm 2} = a_{\pm 1} a_{\pm 1} = \frac{1}{4}$. The terms for $c_{\pm 1}$ all contain $a_0 = 0$, so $c_{\pm 1} = 0$. This agrees with $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta = \frac{1}{2} + \frac{1}{4}e^{j2\theta} + \frac{1}{4}e^{-j2\theta}$.

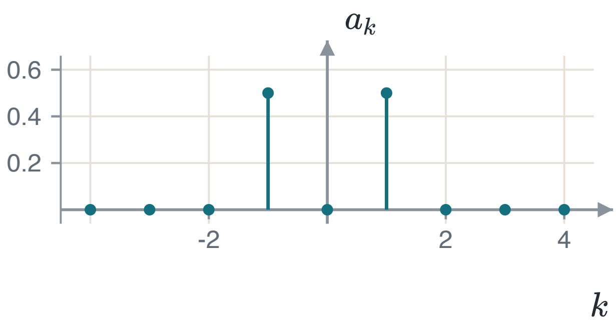

FIGURE 4.55 $\cos(\omega_0 t)$: $a_{\pm 1} = \frac{1}{2}$.

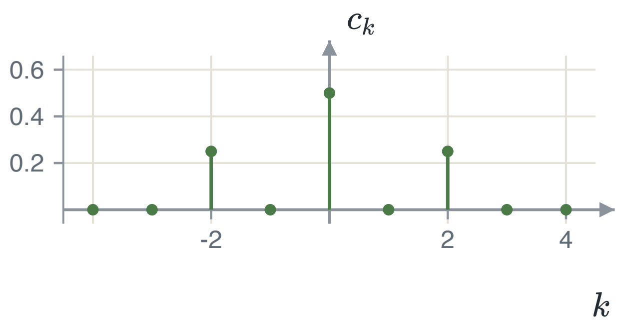

FIGURE 4.56 $\cos^2(\omega_0 t)$: the pair convolved with itself.

Differentiation and integration

Differentiate the synthesis equation term by term. The derivative of $e^{jk\omega_0 t}$ is $jk\omega_0\, e^{jk\omega_0 t}$:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \sum_{k=-\infty}^{\infty} a_k \frac{\mathrm{d}}{\mathrm{d}t} e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} jk\omega_0\, a_k\, e^{jk\omega_0 t}.$$

So $\mathrm{d}x/\mathrm{d}t \leftrightarrow jk\omega_0 a_k$. The factor grows with k, so differentiation raises the high harmonics. For example, $x(t) = \sin(\pi t)$ has $\omega_0 = \pi$ and $a_1 = \frac{1}{2j}$. The derivative has $jk\omega_0 a_1 = j\pi \cdot \frac{1}{2j} = \frac{\pi}{2}$ at $k = 1$. Directly, $\mathrm{d}x/\mathrm{d}t = \pi\cos(\pi t)$, whose coefficient at $k = 1$ is $\frac{\pi}{2}$.

Integration runs the same rule backwards. Suppose the running integral $y(t) = \int_{-\infty}^{t} x(\tau)\,\mathrm{d}\tau$ is periodic, with coefficients c_k. Then $\mathrm{d}y/\mathrm{d}t = x$, and the differentiation rule gives $jk\omega_0 c_k = a_k$ for every k. At $k = 0$ this reads $0 = a_0$. So a periodic integral needs $a_0 = 0$. For $k \neq 0$, $c_k = a_k/(jk\omega_0)$. The rule does not fix c_0, the mean of y; it depends on the constant of integration.

DIFFERENTIATION AND INTEGRATION

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} \longleftrightarrow jk\omega_0\, a_k, \qquad \int_{-\infty}^{t} x(\tau)\,\mathrm{d}\tau \longleftrightarrow \frac{a_k}{jk\omega_0} \ (k \neq 0), \quad \text{only if } a_0 = 0$$

If $a_0 \neq 0$, the mean integrates to the ramp $a_0 t$, which is not periodic. Remove the mean, integrate the rest, then add $a_0 t$ back: $\int_0^t x\,\mathrm{d}\tau = a_0 t + \int_0^t (x - a_0)\,\mathrm{d}\tau$.

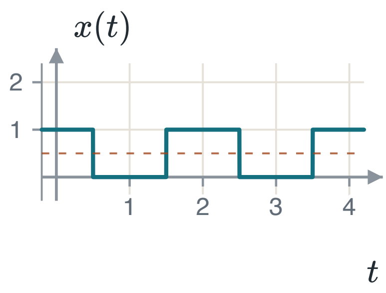

FIGURE 4.57 $x(t)$, $T_0 = 2$, mean $a_0 = 0.5$ (dashed).

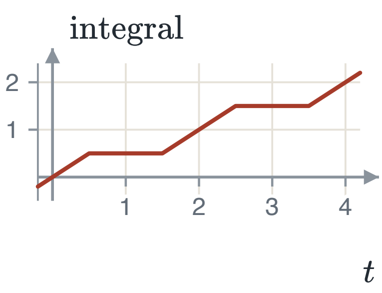

FIGURE 4.58 $\int_0^t x\,\mathrm{d}\tau$ drifts upward.

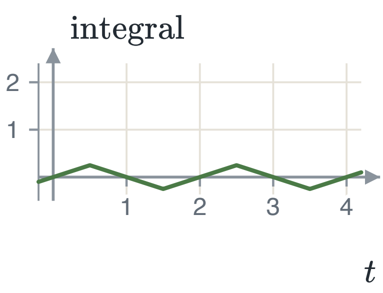

FIGURE 4.59 $\int_0^t (x - a_0)\,\mathrm{d}\tau$ is a periodic triangle.

First difference and running sum

In discrete time the first difference takes the place of the derivative. By linearity and the time shift with $n_0 = 1$,

$$x[n] - x[n-1] \longleftrightarrow a_k - a_k\, e^{-jk(2\pi/N)} = \left(1 - e^{-jk(2\pi/N)}\right) a_k.$$

At $k = 0$ the factor is $1 - e^0 = 0$: a difference of a periodic sequence always has mean zero. The running sum $s[n] = \sum_{r=-\infty}^{n} x[r]$ satisfies $s[n] - s[n-1] = x[n]$. If s is periodic with coefficients c_k, the first-difference rule gives $\left(1 - e^{-jk(2\pi/N)}\right) c_k = a_k$. At $k = 0$ this reads $0 = a_0$, the same condition as for the integral. Otherwise $c_k = a_k / \left(1 - e^{-jk(2\pi/N)}\right)$.

FIRST DIFFERENCE AND RUNNING SUM

$$x[n] - x[n-1] \longleftrightarrow \left(1 - e^{-jk(2\pi/N)}\right) a_k, \qquad \sum_{r=-\infty}^{n} x[r] \longleftrightarrow \frac{a_k}{1 - e^{-jk(2\pi/N)}}, \quad \text{only if } a_0 = 0$$

If $a_0 \neq 0$, each period adds $\sum_{r=\langle N \rangle} x[r] = N a_0$ to the running total, so the sum never repeats.

Take $x[n] = 1, 1, -1, -1$ repeating, with $N = 4$ and $a_0 = \frac{1}{4}(1 + 1 - 1 - 1) = 0$. Its running sum repeats every 4. Now take $x[n] = 1, 1, 1, -1$ repeating, with $a_0 = \frac{1}{4}(1 + 1 + 1 - 1) = \frac{1}{2}$. Its running sum climbs by $N a_0 = 4 \cdot \frac{1}{2} = 2$ every period.

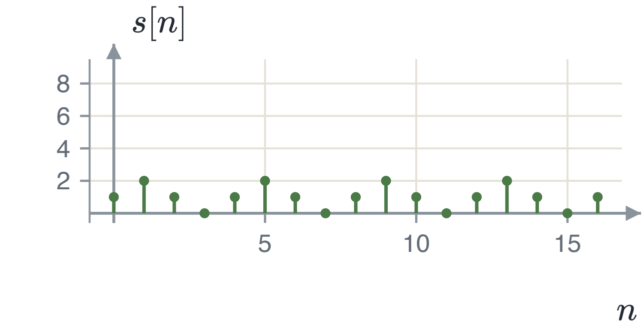

FIGURE 4.60 $x[n] = 1, 1, -1, -1, \ldots$: mean zero, and $s[n] = \sum_{r=0}^{n} x[r]$ repeats.

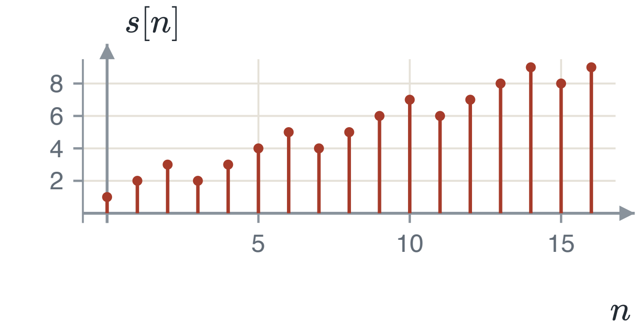

FIGURE 4.61 $x[n] = 1, 1, 1, -1, \ldots$: mean $\frac{1}{2}$, and $s[n]$ climbs by 2 each period.

Parseval's relation

PARSEVAL'S RELATION

$$\frac{1}{T_0}\int_{T_0} |x(t)|^2 \, \mathrm{d}t = \sum_{k=-\infty}^{\infty} |a_k|^2$$

$$\frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2$$

The left side is the average power over one period, with the normalised convention $R = 1\,\Omega$. The right side splits it among the harmonics: $|a_k|^2$ is the power carried by harmonic k alone, and the shares add up to the total.

To derive the continuous-time form, write $|x(t)|^2 = x(t)\,x^*(t)$. Substitute the series for x, and the conjugated series for x^* with a second index m so that the two sums stay separate. Then integrate over one period. The orthogonality integral of Section 4.2 is T_0 when $m = k$ and zero otherwise, so only the terms with $m = k$ survive:

$$\begin{aligned}\frac{1}{T_0}\int_{T_0} x(t)\,x^*(t)\,\mathrm{d}t &= \frac{1}{T_0}\int_{T_0}\Big(\sum_k a_k e^{jk\omega_0 t}\Big)\Big(\sum_m a_m^* e^{-jm\omega_0 t}\Big)\mathrm{d}t \\ &= \sum_k\sum_m a_k\,a_m^*\,\frac{1}{T_0}\int_{T_0} e^{j(k-m)\omega_0 t}\,\mathrm{d}t \\ &= \sum_k a_k\,a_k^* = \sum_k |a_k|^2.\end{aligned}$$

The discrete-time form follows by the same steps, with sums in place of the integral and the discrete orthogonality of Section 4.4. For example, $x(t) = 2\cos(\omega_0 t)$ has $a_{\pm 1} = 1$, so its power is $|a_1|^2 + |a_{-1}|^2 = 2$. Directly, $\frac{1}{T_0}\int_{T_0} 4\cos^2(\omega_0 t)\,\mathrm{d}t = 4 \cdot \frac{1}{2} = 2$.

EXAMPLE 4.14 — PARSEVAL'S RELATION FOR TWO WAVES

GIVEN The rectangular wave with $T_0 = 4T_1$ and the sawtooth with $T_0 = 1$.

FIND The average power of each, in time and from the coefficients, and the share of the rectangular wave's power in its constant term.

METHOD In time, integrate $|x|^2$ over one period. From the coefficients, use Examples 4.7 and 4.8 and the known sum $\sum_{k\geq 1} 1/k^2 = \pi^2/6$. Its odd terms alone add to $\pi^2/6 - \frac{1}{4}\cdot\pi^2/6 = \pi^2/8$, because the even terms are $\sum_{m\geq 1} 1/(2m)^2 = \frac{1}{4}\sum_{m\geq 1} 1/m^2$.

SOLUTION Rectangular wave, in time: $|x|^2 = 1$ on $|t| < T_1$ and 0 elsewhere, so the power is $\frac{1}{T_0}\int_{-T_1}^{T_1} 1\,\mathrm{d}t = \frac{2T_1}{T_0} = \frac{1}{2}$. From the coefficients, $a_0 = \frac{1}{2}$, $a_k^2 = 1/(\pi^2 k^2)$ for odd k, and $a_k = 0$ for even $k \neq 0$. The negative k repeat the positive ones:

$$a_0^2 + 2\sum_{k\text{ odd}\geq 1} \frac{1}{\pi^2 k^2} = \frac{1}{4} + \frac{2}{\pi^2}\cdot\frac{\pi^2}{8} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Sawtooth, in time:

$$\int_{-1/2}^{1/2} t^2\,\mathrm{d}t = \left[\frac{t^3}{3}\right]_{-1/2}^{1/2} = \frac{1}{24} + \frac{1}{24} = \frac{1}{12}.$$

From the coefficients, $|a_k|^2 = 1/(4\pi^2 k^2)$ and $a_0 = 0$:

$$2\sum_{k\geq 1} \frac{1}{4\pi^2 k^2} = \frac{1}{2\pi^2}\cdot\frac{\pi^2}{6} = \frac{1}{12}.$$

The constant term of the rectangular wave carries $a_0^2 = \frac{1}{4}$ of the total $\frac{1}{2}$, which is one half.

CHECK Both waves give the same power in time and from the coefficients.

Parseval's relation also measures truncation error. By linearity, $e_N = x - x_N$ has coefficients $a_k - a_k = 0$ for $|k| \leq N$ and $a_k - 0 = a_k$ for $|k| > N$. Parseval's relation applied to e_N gives

$$\mathrm{MSE} = \frac{1}{T_0}\int_{T_0} |e_N(t)|^2\,\mathrm{d}t = \sum_{|k|>N} |a_k|^2.$$

The mean-square error of a partial sum is exactly the power in the dropped harmonics. For the rectangular wave with $T_0 = 4T_1$ and $N = 3$, the kept power is $\frac{1}{4} + 2\left(\frac{1}{\pi^2} + \frac{1}{9\pi^2}\right) = \frac{1}{4} + \frac{20}{9\pi^2} = 0.4752$, so the MSE is $0.5 - 0.4752 = 0.025$, the first entry of the table in Section 4.3.

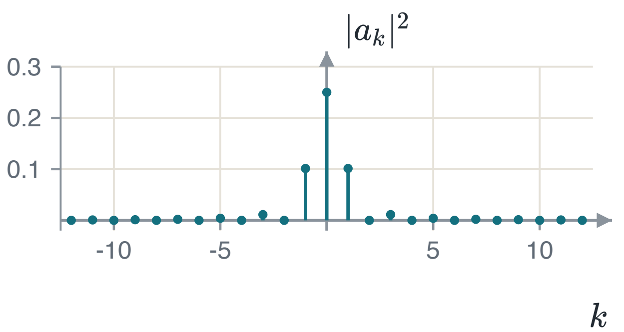

FIGURE 4.62 Power per harmonic of the rectangular wave with $T_0 = 4T_1$. The stems add to 0.5.

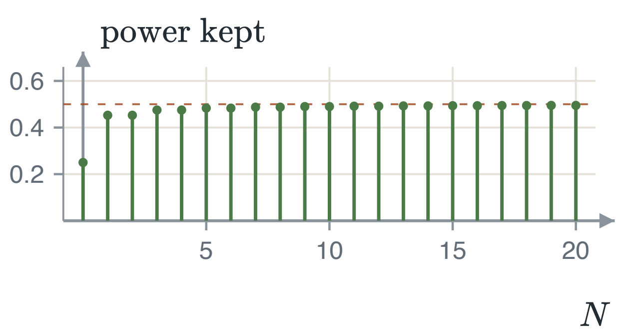

FIGURE 4.63 Power in the harmonics $|k| \leq N$. It reaches the total 0.5 (dashed) only in the limit.

4.6 A periodic input through an LTI system

Apply the eigenfunction property of Section 4.1 to each term of the synthesis equation. The term $a_k e^{jk\omega_0 t}$ is a complex exponential with $s = jk\omega_0$, so the system multiplies it by $H(jk\omega_0)$. The system is linear, so the outputs of all the terms add:

ONE MULTIPLICATION PER HARMONIC

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \longrightarrow y(t) = \sum_{k=-\infty}^{\infty} \underbrace{a_k H(jk\omega_0)}_{b_k} e^{jk\omega_0 t}$$

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} \longrightarrow y[n] = \sum_{k=\langle N\rangle} a_k H(e^{jk\omega_0})\, e^{jk\omega_0 n}$$

The output has the period of the input. The frequency response describes the system. For a periodic input, calculate one product at each harmonic instead of a convolution.

Each b_k is a product of complex numbers, so magnitudes multiply and phases add: $|b_k| = |a_k|\,|H(jk\omega_0)|$ and $\angle b_k = \angle a_k + \angle H(jk\omega_0)$. For example, $a_1 = 1$ and $H(j\omega_0) = e^{-j\pi/4}$ give $b_1 = e^{-j\pi/4}$.

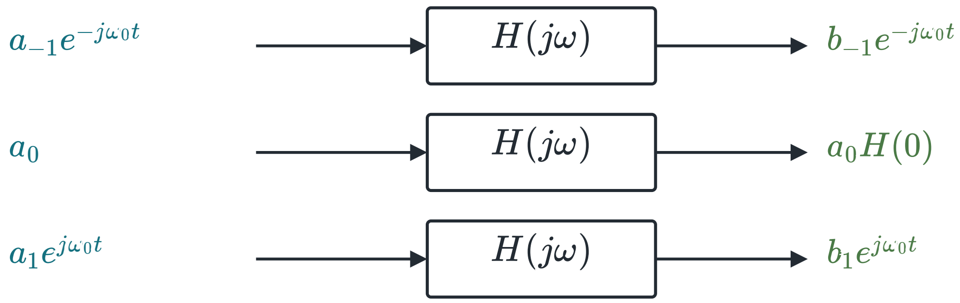

FIGURE 4.64 Each harmonic is an eigenfunction. It passes through on its own and comes out scaled by H at its own frequency.

Putting a real output back together

The result above is a sum of complex terms. For real x, $a_{-k} = a_k^*$. For real h, conjugating $H(j\omega) = \int h(\tau)e^{-j\omega\tau}\,\mathrm{d}\tau$ conjugates only the exponential, so $H^*(j\omega) = \int h(\tau)e^{+j\omega\tau}\,\mathrm{d}\tau = H(-j\omega)$. Together, $b_{-k} = a_{-k}H(-jk\omega_0) = a_k^* H^*(jk\omega_0) = b_k^*$. Pair each k with $-k$, write $b_k = |b_k|e^{j\angle b_k}$, and use $e^{j\theta} + e^{-j\theta} = 2\cos\theta$:

$$\begin{aligned} b_k e^{jk\omega_0 t} + b_{-k}e^{-jk\omega_0 t} &= |b_k|e^{j\angle b_k}e^{jk\omega_0 t} + |b_k|e^{-j\angle b_k}e^{-jk\omega_0 t} \\ &= |b_k|\left[e^{j(k\omega_0 t + \angle b_k)} + e^{-j(k\omega_0 t + \angle b_k)}\right] \\ &= 2|b_k|\cos\bigl(k\omega_0 t + \angle b_k\bigr). \end{aligned}$$

CONJUGATE-PAIR REASSEMBLY

$$y(t) = b_0 + \sum_{k=1}^{\infty} 2|b_k| \cos\big(k\omega_0 t + \angle b_k\big)$$

b_0 has no partner, so it carries no factor of two. This is the amplitude–phase form of Section 4.2 applied to the output.

TWO SLIPS

The amplitude of the k-th cosine is $2|b_k|$, because both members of the pair contribute. Using $|b_k|$ halves every amplitude. The phase is $\angle b_k$, the one for **positive** k; using $\angle b_{-k}$ flips every phase. For example, $b_1 = 0.6e^{j0.5}$ with $\omega_0 = \pi$ gives $1.2\cos(\pi t + 0.5)$, not $0.6\cos(\pi t + 0.5)$. Test any reassembly by setting $H = 1$: it must return the input exactly.

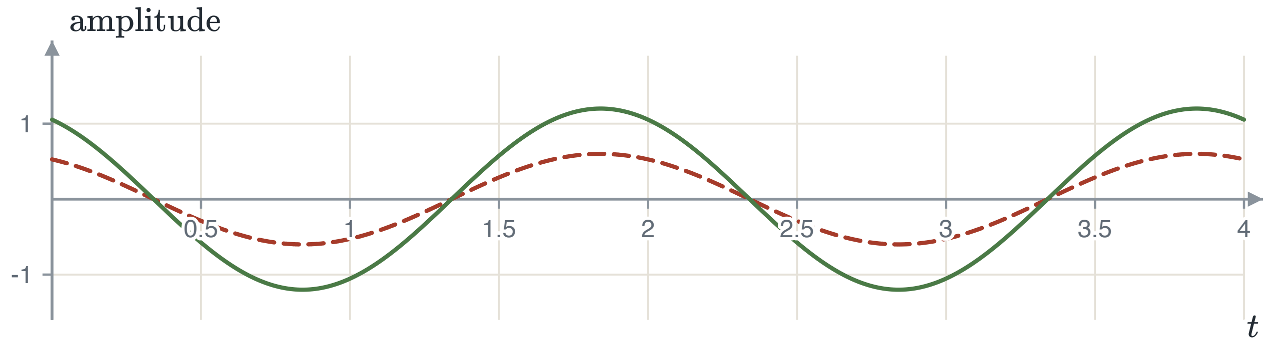

FIGURE 4.65 $b_1 = 0.6e^{j0.5}$ with $\omega_0 = \pi$. The pair $k = \pm 1$ adds to $2|b_1|\cos(\pi t + 0.5)$, amplitude 1.2 (solid). One term alone gives half (dashed).

EXAMPLE 4.15 – LOW-PASS FILTERING

GIVEN $x(t) = 1 + \cos(\pi t) + \sin(2\pi t) + \cos\big(3\pi t + \frac{\pi}{3}\big)$ into the system with $h(t) = e^{-t}u(t)$.

FIND The output signal.

METHOD The input is periodic and the system is LTI, so harmonic multiplication applies. Take $H(j\omega)$ from Example 4.1. Find T_0 and a_k from x. Form $b_k = a_k H(jk\omega_0)$ and reassemble each conjugate pair.

SOLUTION Example 4.1 gives $H(j\omega) = 1/(1 + j\omega)$, so $|H(j\omega)| = 1/\sqrt{1+\omega^2}$ and $\angle H(j\omega) = -\arctan\omega$. $|H|$ falls from 1 at $\omega = 0$ towards 0: a low-pass system. Next the input. The component periods are $2\pi/\pi = 2$, $2\pi/(2\pi) = 1$ and $2\pi/(3\pi) = 2/3$ s. This is case (c) of Example 4.4, so $T_0 = 2$ s and $\omega_0 = \pi$ rad/s. The three frequencies are $1\omega_0$, $2\omega_0$ and $3\omega_0$. Euler's relations give the coefficients for positive k:

$$\cos(\pi t) \to a_1 = \tfrac{1}{2},$$
$$\sin(2\pi t) = \tfrac{1}{2j}e^{j2\pi t} - \tfrac{1}{2j}e^{-j2\pi t} \to a_2 = \tfrac{1}{2j} = \tfrac{1}{2}e^{-j\pi/2},$$
$$\cos\big(3\pi t + \tfrac{\pi}{3}\big) \to a_3 = \tfrac{1}{2}e^{j\pi/3},$$

with $a_0 = 1$ and $a_{-k} = a_k^*$. The values of H at the three harmonics are

$$|H(j\pi)| = \frac{1}{\sqrt{1+\pi^2}} = 0.3033, \qquad \angle H(j\pi) = -\arctan\pi = -1.2626,$$
$$|H(j2\pi)| = \frac{1}{\sqrt{1+4\pi^2}} = 0.1572, \quad \angle H(j2\pi) = -\arctan 2\pi = -1.4130,$$
$$|H(j3\pi)| = \frac{1}{\sqrt{1+9\pi^2}} = 0.1055, \quad \angle H(j3\pi) = -\arctan 3\pi = -1.4651.$$

Now one product per harmonic, $b_k = a_k H(jk\pi)$: magnitudes multiply and phases add.

$$b_0 = a_0 H(j0) = 1\cdot 1 = 1,$$
$$b_1 = \tfrac{1}{2}(0.3033)\,e^{-j1.2626} = 0.1517e^{-j1.263},$$
$$b_2 = \tfrac{1}{2}(0.1572)\,e^{j(-\pi/2-1.4130)} = 0.0786e^{-j2.984},$$
$$b_3 = \tfrac{1}{2}(0.1055)\,e^{j(\pi/3-1.4651)} = 0.0528e^{-j0.418}.$$

Finally reassemble each pair into a cosine of amplitude $2|b_k|$ and phase $\angle b_k$:

$$y(t) = 1 + 0.303\cos(\pi t - 1.263) + 0.157\cos(2\pi t - 2.984)$$
$$+ 0.106\cos(3\pi t - 0.418).$$

CHECK — Each amplitude is $2|b_k|$: $2\cdot 0.1517 = 0.303$, $2\cdot 0.0786 = 0.157$, $2\cdot 0.0528 = 0.106$. The average is $b_0 = 1$, since $H(0) = 1$. The third harmonic falls from amplitude 1 to 0.106, about one tenth. The output ranges from about 0.615 to 1.417; half the amplitudes would give only 0.81 to 1.21.

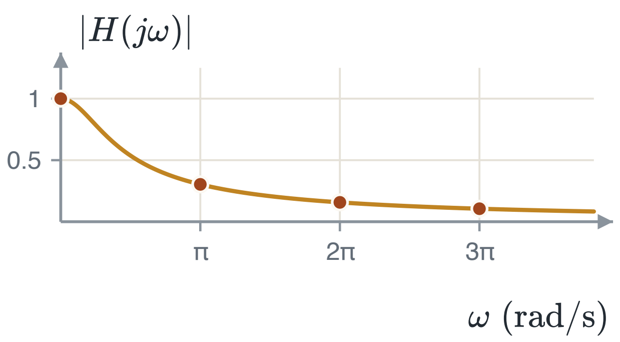

FIGURE 4.66 $|H(j\omega)|$ with the four input harmonics marked at $k\omega_0$, $\omega_0 = \pi$.

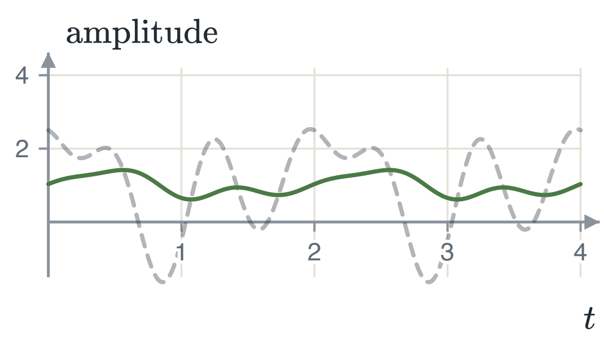

FIGURE 4.67 $x(t)$ (dashed) and $y(t)$ (solid): the average stays at 1 and the fast wiggles shrink.

EXAMPLE 4.16 — HIGH-PASS FILTERING

GIVEN — The same $x(t)$, now into a system with $H(j\omega) = \dfrac{j\omega}{1+j\omega}$.

FIND — The output signal.

METHOD — Use the same harmonic multiplication as in Example 4.15. Find the magnitude and phase of the new H, compute each b_k, and reassemble the pairs.

SOLUTION The magnitude of a quotient is the quotient of the magnitudes, and its phase is the difference of the phases. The numerator $j\omega$ has magnitude $|\omega|$ and phase $+\pi/2$ for $\omega > 0$, $-\pi/2$ for $\omega < 0$. The denominator $1 + j\omega$ has magnitude $\sqrt{1+\omega^2}$ and phase $\arctan\omega$:

$$|H(j\omega)| = \frac{|\omega|}{\sqrt{1+\omega^2}}, \qquad \angle H(j\omega) = \frac{\pi}{2}\,\mathrm{sgn}(\omega) - \arctan\omega.$$

First, $H(j0) = 0/(1+0) = 0$, so $b_0 = a_0 H(j0) = 0$ for every input: removing the average is what a high-pass system does. For $k \geq 1$, with the values of $\sqrt{1+k^2\pi^2}$ and $\arctan(k\pi)$ from Example 4.15,

$$\begin{aligned}
|H(j\pi)| &= \frac{3.1416}{3.2969} = 0.9529, & \angle H(j\pi) &= 1.5708 - 1.2626 = 0.3082,\\
|H(j2\pi)| &= \frac{6.2832}{6.3623} = 0.9876, & \angle H(j2\pi) &= 1.5708 - 1.4130 = 0.1578,\\
|H(j3\pi)| &= \frac{9.4248}{9.4777} = 0.9944, & \angle H(j3\pi) &= 1.5708 - 1.4651 = 0.1057.
\end{aligned}$$

The input coefficients are as before: $a_1 = \frac{1}{2}$, $a_2 = \frac{1}{2}e^{-j\pi/2}$, $a_3 = \frac{1}{2}e^{j\pi/3}$. Multiply the magnitudes and add the phases:

$$\begin{aligned}
b_1 &= \tfrac{1}{2}(0.9529)\,e^{j0.3082} = 0.4764e^{j0.308},\\
b_2 &= \tfrac{1}{2}(0.9876)\,e^{j(-\pi/2+0.1578)} = 0.4938e^{-j1.413},\\
b_3 &= \tfrac{1}{2}(0.9944)\,e^{j(\pi/3+0.1057)} = 0.4972e^{j1.153}.
\end{aligned}$$

Pair k with $-k$ as before:

$$y(t) = 0.953\cos(\pi t + 0.308) + 0.988\cos(2\pi t - 1.413) + 0.994\cos(3\pi t + 1.153).$$

There is no constant term, because $b_0 = 0$.

CHECK $|H(jk\pi)|$ is close to 1 for $k = 1, 2, 3$. So y should look like x with its average of 1 removed, and the output amplitudes must be close to 1. Amplitudes of 0.48, 0.49 and 0.50 would show that the factor of two is missing. Phases of -0.308, $+1.413$ and -1.153 would show that they were read from b_{-k}.

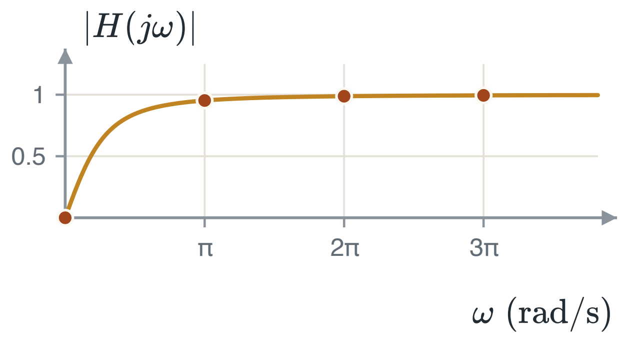

FIGURE 4.68 $|H(j\omega)|$ of the high-pass system, with the same four harmonics.

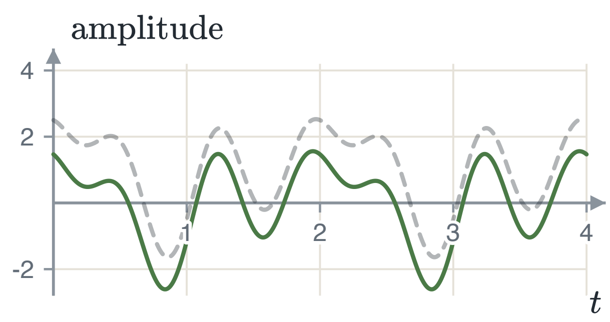

FIGURE 4.69 $x(t)$ (dashed) and $y(t)$ (solid): the harmonics pass almost unchanged and the average drops to 0.

Ideal frequency-selective filters

A frequency-selective filter passes some bands of frequency and stops others. The ideal low-pass filter passes every frequency below a cutoff unchanged and removes every frequency above it:

IDEAL LOW-PASS FILTER

$$H(j\omega) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & |\omega| > \omega_c \end{cases} \quad \Longrightarrow \quad b_k = a_k H(jk\omega_0) = \begin{cases} a_k, & |k|\omega_0 < \omega_c \\ 0, & |k|\omega_0 > \omega_c \end{cases}$$

$|\omega| < \omega_c$ is the passband, the rest is the stopband, and ω_c in rad/s is the cutoff frequency. A real filter can only approach this shape. For a periodic input, the output is a partial sum of the input series, so beside each jump it shows the Gibbs ripple.

EXAMPLE 4.17 – AN IDEAL LOW-PASS FILTER ON A RECTANGULAR WAVE

GIVEN The rectangular wave with $T_0 = 2$ and $T_1 = 0.5$, so $\omega_0 = \pi$ rad/s, into an ideal low-pass filter with $\omega_c = 3.5\pi$ rad/s. Then the same wave with $\omega_c = 2.5\pi$ rad/s.

FIND Both outputs.

METHOD Here $T_0 = 4T_1$, so Example 4.7 gives $a_0 = \frac{1}{2}$ and $a_k = \sin(\pi k/2)/(\pi k)$. Keep the harmonics with $|k|\pi < \omega_c$. The coefficients are real, so each pair gives $2a_k \cos(k\pi t)$.

SOLUTION With $\omega_c = 3.5\pi$ the passband keeps $|k| \leq 3$. The kept values are $a_0 = \frac{1}{2}$, $a_{\pm 1} = 1/\pi$, $a_{\pm 2} = 0$ and $a_{\pm 3} = -1/(3\pi)$. So

$$y(t) = \frac{1}{2} + \frac{2}{\pi}\cos \pi t - \frac{2}{3\pi}\cos 3\pi t.$$

With $\omega_c = 2.5\pi$ the passband keeps $|k| \leq 2$, and $a_{\pm 2} = 0$, so only one cosine remains besides the constant:

$$y(t) = \frac{1}{2} + \frac{2}{\pi}\cos \pi t.$$

CHECK At the jump $t = 0.5$, $\cos(\pi/2) = \cos(3\pi/2) = 0$, so both outputs equal $\frac{1}{2}$, the midpoint of the jump from 1 to 0. At $t = 0$ the first output is $\frac{1}{2} + \frac{2}{\pi} - \frac{2}{3\pi} = \frac{1}{2} + \frac{4}{3\pi} = 0.924$, close to $x(0) = 1$.

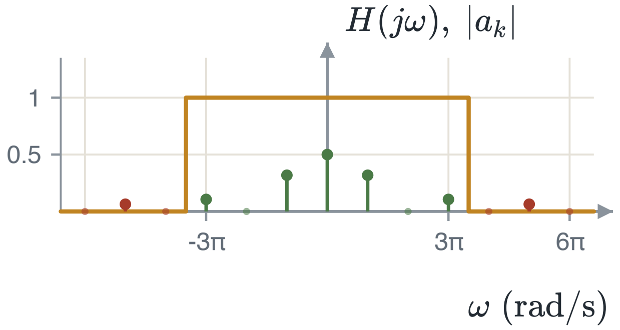

FIGURE 4.70 $\omega_c = 3.5\pi$: the harmonics inside the passband pass unchanged; the rest are removed.

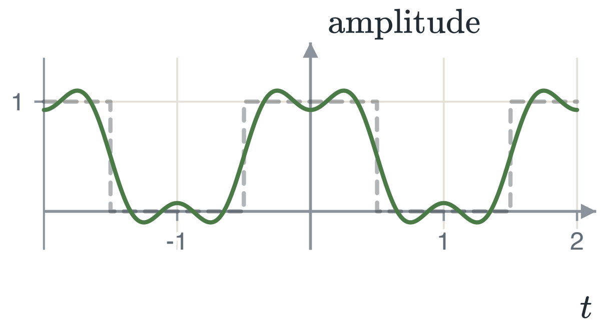

FIGURE 4.71 The output (solid) is the partial sum with $N = 3$ of the wave (dashed).

Frequency-shaping filters

A frequency-shaping filter changes the relative size of the harmonics instead of passing or stopping whole bands. An audio equaliser is one: it raises or lowers the bass and the treble. The differentiator, $H(j\omega) = j\omega$, is a simple example. It gives $b_k = jk\omega_0\, a_k$, which is the differentiation property of Section 4.5. Its gain $|k|\omega_0$ grows with k, so the high harmonics are raised.

EXAMPLE 4.18 – A DIFFERENTIATOR RAISES THE THIRD HARMONIC

GIVEN $x(t) = \cos t + \frac{1}{9}\cos 3t$, so $\omega_0 = 1$ rad/s, into $H(j\omega) = j\omega$.

FIND The output, and how the size of the third harmonic relative to the first changes.

METHOD Read the a_k with Euler's relation, form $b_k = jk\omega_0 a_k$, and reassemble the pairs.

SOLUTION $a_{\pm 1} = \frac{1}{2}$ and $a_{\pm 3} = \frac{1}{18}$. Then $b_1 = j \cdot 1 \cdot \frac{1}{2} = \frac{1}{2}e^{j\pi/2}$ and $b_3 = j \cdot 3 \cdot \frac{1}{18} = \frac{1}{6}e^{j\pi/2}$. Reassemble with amplitude $2|b_k|$ and phase $\angle b_k$, and use $\cos(\theta + \pi/2) = -\sin\theta$:

$$y(t) = 2 \cdot \tfrac{1}{2}\cos\left(t + \tfrac{\pi}{2}\right) + 2 \cdot \tfrac{1}{6}\cos\left(3t + \tfrac{\pi}{2}\right) = -\sin t - \tfrac{1}{3}\sin 3t.$$

The third harmonic is $\frac{1}{9}$ of the first at the input and $\frac{1}{3}$ at the output. The ratio grew by a factor of 3, the ratio of the gains $|H(j3)|/|H(j1)| = 3/1$.

CHECK Differentiate directly: $\frac{\mathrm{d}}{\mathrm{d}t}\left(\cos t + \frac{1}{9}\cos 3t\right) = -\sin t - \frac{3}{9}\sin 3t = -\sin t - \frac{1}{3}\sin 3t$. In the same way, for $x(t) = \cos 2\pi t + \cos 4\pi t$ the gains are 2π and 4π, so the second harmonic comes out twice as large as the first.

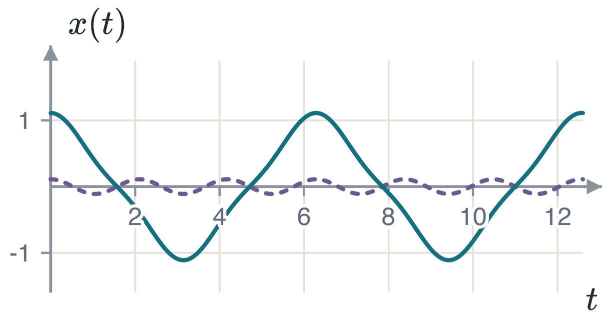

FIGURE 4.72 The input. The dotted trace is its third harmonic, $\frac{1}{9}\cos 3t$.

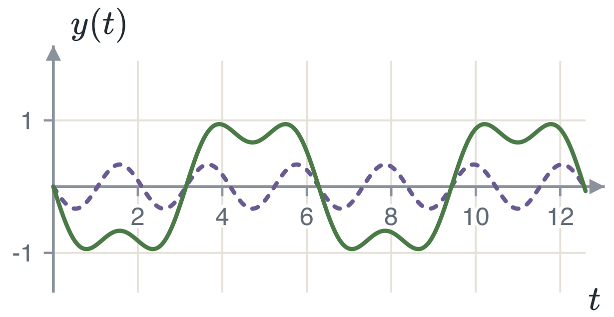

FIGURE 4.73 The output. Its third harmonic, $-\frac{1}{3}\sin 3t$, is relatively three times larger.

Discrete-time filtering

EXAMPLE 4.19 – TWO TWO-TAP FILTERS ON AN IMPULSE TRAIN

GIVEN $x[n] = \sum_{m=-\infty}^{\infty} \delta[n - 4m]$, and the systems $h_1[n] = 0.5\delta[n] - 0.5\delta[n-1]$ and $h_2[n] = 0.5\delta[n] + 0.5\delta[n-1]$.

FIND Both output signals.

METHOD The input is periodic and both systems are LTI, so discrete-time harmonic multiplication applies. Find a_k for the impulse train, compute $b_k = a_k H(e^{jk\omega_0})$ with $\omega_0 = 2\pi/4 = \pi/2$, and reassemble the pairs.

SOLUTION First the input coefficients. One period is $n = 0, 1, 2, 3$ and contains one impulse, at $n = 0$, so

$$a_k = \frac{1}{4}\sum_{n=0}^{3} x[n]\, e^{-jk(\pi/2)n} = \frac{1}{4}\cdot 1 \cdot e^{-jk(\pi/2)\cdot 0} = \frac{1}{4} \quad \text{for every } k.$$

The spectrum is flat. One period of coefficients is $k = -1, 0, 1, 2$. Next the frequency responses, from $H(e^{j\omega}) = \sum_n h[n]e^{-j\omega n}$. Factor out $e^{-j\omega/2}$ as in Example 4.2:

$$H_1(e^{j\omega}) = 0.5\big(1 - e^{-j\omega}\big) = 0.5e^{-j\omega/2}\big(e^{j\omega/2} - e^{-j\omega/2}\big) = je^{-j\omega/2}\sin(\omega/2),$$
$$H_2(e^{j\omega}) = 0.5\big(1 + e^{-j\omega}\big) = e^{-j\omega/2}\cos(\omega/2).$$

So $|H_1| = |\sin(\omega/2)|$ is 0 at $\omega = 0$ and 1 at $\omega = \pi$: a high-pass system. $|H_2| = |\cos(\omega/2)|$ is the reverse: a low-pass system. Evaluate H_1 at the harmonics $\omega = 0, \pi/2, \pi$, using $e^{-j\pi/2} = -j$ and $e^{-j\pi} = -1$:

$$H_1(e^{j0}) = 0.5(1 - 1) = 0,$$
$$H_1(e^{j\pi/2}) = 0.5(1 + j) = 0.5\sqrt{2}\, e^{j\pi/4} = 0.7071e^{j\pi/4},$$
$$H_1(e^{j\pi}) = 0.5(1 + 1) = 1.$$

Multiply by $a_k = \frac{1}{4}$: $b_0 = 0$, $b_1 = 0.25(0.7071)e^{j\pi/4} = 0.1768e^{j\pi/4}$ and $b_2 = 0.25$. Also $b_{-1} = b_3 = b_1^* = 0.1768e^{-j\pi/4}$. The pair $k = \pm 1$ gives a cosine of amplitude $2|b_1| = 2 \cdot \frac{\sqrt{2}}{8} = \frac{\sqrt{2}}{4} = 0.354$ and phase $\pi/4$. The term $k = 2$ has $e^{j2(\pi/2)n} = e^{j\pi n} = (-1)^n$. So

$$y_1[n] = 0.354\cos\left(\frac{\pi}{2}n + \frac{\pi}{4}\right) + 0.25(-1)^n.$$

For h_2 the same three evaluations give

$$H_2(e^{j0}) = 0.5(1 + 1) = 1,$$
$$H_2(e^{j\pi/2}) = 0.5(1 - j) = 0.7071e^{-j\pi/4},$$
$$H_2(e^{j\pi}) = 0.5(1 - 1) = 0,$$

so $b_0 = 0.25$, $b_1 = 0.1768e^{-j\pi/4}$ and $b_2 = 0$, and

$$y_2[n] = 0.25 + 0.354\cos\left(\frac{\pi}{2}n - \frac{\pi}{4}\right).$$

CHECK Apply h_1 directly. $y_1[n] = 0.5x[n] - 0.5x[n-1]$ turns each impulse into 0.5 followed by -0.5, so one period is $0.5, -0.5, 0, 0$. The formula gives the same: at $n = 0$, $0.354\cos(\pi/4) + 0.25 = 0.25 + 0.25 = 0.5$; at $n = 1$, $0.354\cos(3\pi/4) - 0.25 = -0.25 - 0.25 = -0.5$; at $n = 2$, $0.354\cos(5\pi/4) + 0.25 = 0$; at $n = 3$, $0.354\cos(7\pi/4) - 0.25 = 0$. For h_2 each impulse spreads into two samples of 0.5: at $n = 0$, $0.25 + 0.354\cos(-\pi/4) = 0.5$, and at $n = 2$, $0.25 + 0.354\cos(3\pi/4) = 0$.

⚠ THE TERM AT $k = N/2$

With $N = 4$, the term $k = 2$ is its own partner: $e^{j\pi n} = (-1)^n$ is real, and $k = -2$ lies in the same period as $k = 2$. It enters once, as $b_2(-1)^n$, without a factor of two. The high-pass system removes b_0 and keeps b_2. The low-pass system keeps b_0 and removes b_2. Both frequency responses repeat every 2π in ω.

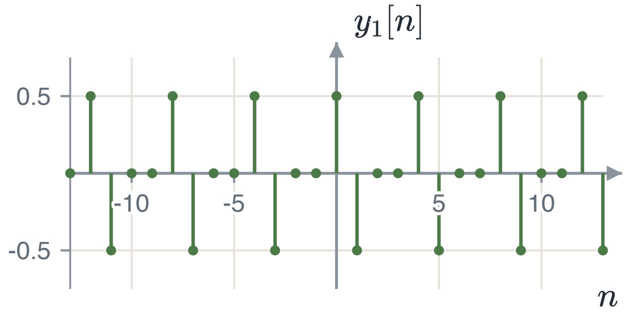

FIGURE 4.74 High-pass h_1: zero average.

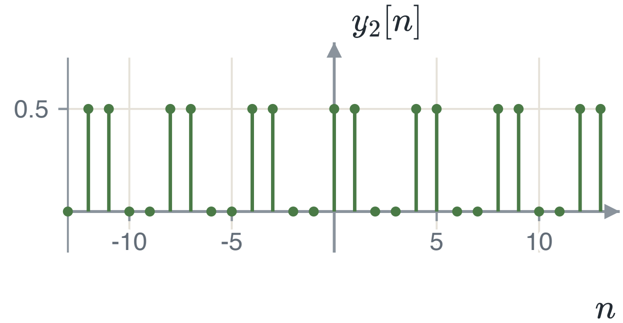

FIGURE 4.75 Low-pass h_2: the average remains.

A first-order recursive filter

The two-tap filters use inputs only: they are **nonrecursive**. A **recursive** filter also uses its own previous output. The simplest one is

$$y[n] - a\,y[n-1] = x[n], \qquad |a| < 1.$$

Its frequency response follows from the eigenfunction property. Put in $x[n] = e^{j\omega n}$ and $y[n] = H(e^{j\omega})\,e^{j\omega n}$, then divide by $e^{j\omega n}$:

$$\begin{aligned} He^{j\omega n} - a\,He^{j\omega(n-1)} &= e^{j\omega n} \\ H\left(1 - ae^{-j\omega}\right) &= 1 \\ H(e^{j\omega}) &= \frac{1}{1 - ae^{-j\omega}}. \end{aligned}$$

The condition $|a| < 1$ makes the system stable. From initial rest, the impulse response satisfies $h[0] = 1$ and $h[n] = a\,h[n-1]$ for $n \geq 1$, so $h[n] = a^n u[n]$. Its absolute sum is a geometric series with ratio $|a| < 1$: $\sum_{n=0}^{\infty} |a|^n = 1/(1 - |a|)$, which is finite. The same h gives the frequency response again, as a geometric series with ratio $ae^{-j\omega}$, where $|ae^{-j\omega}| = |a| < 1$:

$$H(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} \left(ae^{-j\omega}\right)^n = \frac{1}{1 - ae^{-j\omega}}.$$

The gain at the two ends of the band shows what the filter does. At $\omega = 0$, $e^{-j0} = 1$, and at $\omega = \pi$, $e^{-j\pi} = -1$, so

$$H(e^{j0}) = \frac{1}{1-a}, \qquad H(e^{j\pi}) = \frac{1}{1+a}, \qquad |H(e^{j\omega})| = \frac{1}{\sqrt{1 - 2a\cos\omega + a^2}}.$$

The last form uses $|1 - ae^{-j\omega}|^2 = (1 - a\cos\omega)^2 + (a\sin\omega)^2 = 1 - 2a\cos\omega + a^2$. With $a > 0$ the gain is largest at $\omega = 0$: the filter is low-pass. With $a < 0$ it is largest at $\omega = \pi$: the filter is high-pass. At

$a = 0$ every gain is 1. For $a = 0.5$, $|H(e^{j\pi})| = 1/(1+0.5) = \frac{2}{3}$.

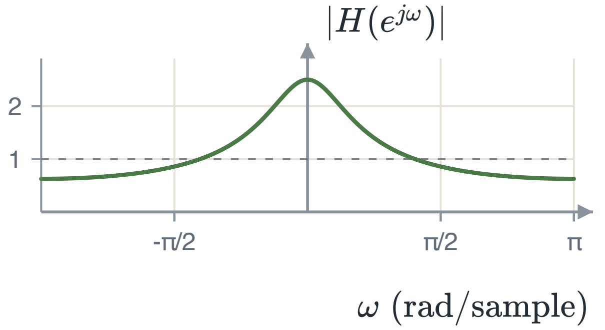

FIGURE 4.76 $a = 0.6$: gain 2.5 at $\omega = 0$, low-pass.

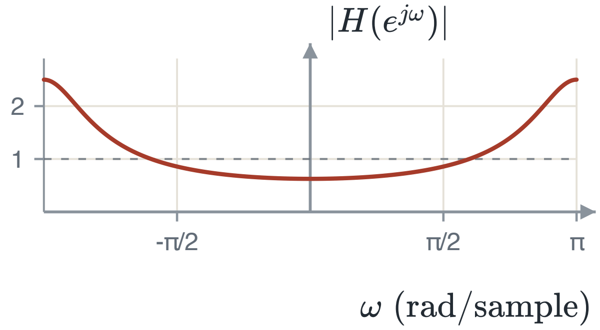

FIGURE 4.77 $a = -0.6$: gain 2.5 at $\omega = \pm\pi$, high-pass.

EXAMPLE 4.20 — A RECURSIVE FILTER ON AN IMPULSE TRAIN

GIVEN $x[n] = \sum_{m=-\infty}^{\infty} \delta[n-4m]$ into $y[n] - 0.5\,y[n-1] = x[n]$.

FIND The output $y[n]$.

METHOD Take $a_k = \frac{1}{4}$ and $\omega_0 = \pi/2$ from Example 4.19. Evaluate $H(e^{jk\pi/2})$ for $k = 0, 1, 2$, form $b_k = a_k H(e^{jk\pi/2})$, and reassemble.

SOLUTION With $a = 0.5$, $H(e^{j\omega}) = 1/(1 - 0.5e^{-j\omega})$. At the harmonics, using $e^{-j\pi/2} = -j$ and $e^{-j\pi} = -1$:

$$H(e^{j0}) = \frac{1}{1-0.5} = 2,$$
$$H(e^{j\pi/2}) = \frac{1}{1+0.5j},$$
$$H(e^{j\pi}) = \frac{1}{1+0.5} = \frac{2}{3}.$$

For the middle value, $|1+0.5j| = \sqrt{1.25} = 1.118$ and $\angle(1+0.5j) = \arctan 0.5 = 0.464$ rad, so $H(e^{j\pi/2}) = 0.894e^{-j0.464}$. Then

$$b_0 = \tfrac{1}{4}\cdot 2 = \tfrac{1}{2},$$
$$b_1 = \frac{1/4}{1+0.5j} = \frac{0.25}{1.118}e^{-j0.464} = 0.2236e^{-j0.464},$$
$$b_2 = \tfrac{1}{4}\cdot\tfrac{2}{3} = \tfrac{1}{6}.$$

Also $b_{-1} = b_3 = b_1^*$, because x and the filter are real. The pair $k = \pm 1$ gives amplitude $2|b_1| = 0.447$. The term at $k = N/2 = 2$ enters once, as $b_2(-1)^n$:

$$y[n] = \tfrac{1}{2} + 0.447\cos\left(\frac{\pi}{2}n - 0.464\right) + \tfrac{1}{6}(-1)^n.$$

CHECK Run the difference equation over one period. For $n = 1, 2, 3$ the input is zero, so $y[1] = 0.5y[0]$, $y[2] = 0.25y[0]$ and $y[3] = 0.125y[0]$. At $n = 4$ the next impulse arrives and $y[4] = y[0]$ by periodicity: $y[0] = 1 + 0.5y[3] = 1 + 0.0625y[0]$, which gives $y[0] = 1/0.9375 = 16/15 = 1.067$. The formula gives $0.5 + 0.447\cos(-0.464) + 0.167 =$

$0.5 + 0.4 + 0.167 = 1.067$, since $0.447\cos(0.464) = \sqrt{0.2} \cdot 2/\sqrt{5} = 0.4$. At $n = 1$ it gives $0.5 + 0.447\sin(0.464) - 0.167 = 0.5 + 0.2 - 0.167 = 0.533 = y[0]/2$.

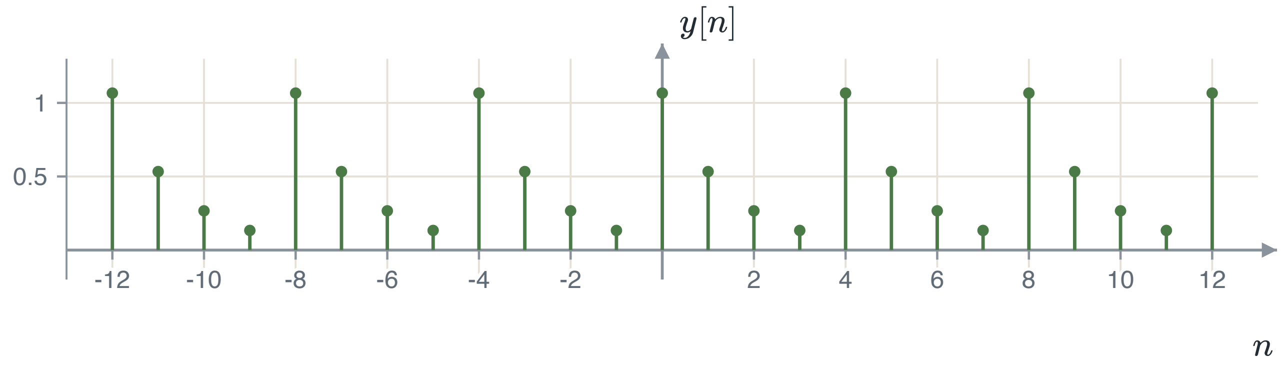

FIGURE 4.78 The output of Example 4.20. Each impulse leaves a tail 0.5^n that reaches into the next period, because the output feeds back. Each two-tap response of Example 4.19 ended after two samples.

4.7 Summary

The two tables collect the properties of Section 4.5. To use them, recognise the signal as a known one with an operation applied. Look up the known coefficients, apply the rows in order, and check a_0 against the mean of the signal.

TABLE 4.2 Properties of the continuous-time Fourier series.

PROPERTY	CONTINUOUS TIME, PERIOD T_0, $\omega_0 = 2\pi/T_0$
Linearity	$Ax(t) + By(t) \;\leftrightarrow\; Aa_k + Bb_k$
Time shift	$x(t - t_0) \;\leftrightarrow\; a_k e^{-jk\omega_0 t_0}$
Frequency shift	$e^{jM\omega_0 t}x(t) \;\leftrightarrow\; a_{k-M}$, M an integer
Conjugation	$x^*(t) \;\leftrightarrow\; a^*_{-k}$
Time reversal	$x(-t) \;\leftrightarrow\; a_{-k}$
Time scaling	$x(\alpha t) \;\leftrightarrow\; a_k$, $\alpha > 0$, period T_0/α
Periodic convolution	$\int_{T_0} x(\tau)y(t-\tau)\,\mathrm{d}\tau \;\leftrightarrow\; T_0 a_k b_k$
Multiplication	$x(t)y(t) \;\leftrightarrow\; \sum_{\ell=-\infty}^{\infty} a_\ell b_{k-\ell}$
Differentiation	$\mathrm{d}x/\mathrm{d}t \;\leftrightarrow\; jk\omega_0 a_k$
Integration	$\int_{-\infty}^{t} x(\tau)\,\mathrm{d}\tau \;\leftrightarrow\; a_k/(jk\omega_0)$, only if $a_0 = 0$
Real signal	$a_{-k} = a_k^*$: $\lvert a_k\rvert$ even, $\angle a_k$ odd
Real and even	a_k real and even
Real and odd	a_k purely imaginary and odd
Even and odd parts	$\mathcal{E}\mathrm{v}\{x\} \leftrightarrow \mathrm{Re}\{a_k\}$, $\mathcal{O}\mathrm{dd}\{x\} \leftrightarrow j\,\mathrm{Im}\{a_k\}$, for real x
Parseval	$\frac{1}{T_0}\int_{T_0} \lvert x(t)\rvert^2\,\mathrm{d}t = \sum_k \lvert a_k\rvert^2$

TABLE 4.3 Properties of the discrete-time Fourier series.

PROPERTY	DISCRETE TIME, PERIOD N, $\omega_0 = 2\pi/N$
Linearity	$Ax[n] + By[n] \;\leftrightarrow\; Aa_k + Bb_k$
Time shift	$x[n-n_0] \;\leftrightarrow\; a_k e^{-jk(2\pi/N)n_0}$
Frequency shift	$e^{jM(2\pi/N)n}x[n] \;\leftrightarrow\; a_{k-M}$, M an integer
Conjugation	$x^*[n] \;\leftrightarrow\; a^*_{-k}$
Time reversal	$x[-n] \;\leftrightarrow\; a_{-k}$
Time scaling	$x_{(m)}[n] \;\leftrightarrow\; a_k/m$, period mN
Periodic convolution	$\sum_{r=\langle N\rangle} x[r]y[n-r] \;\leftrightarrow\; Na_kb_k$
Multiplication	$x[n]y[n] \;\leftrightarrow\; \sum_{\ell=\langle N\rangle} a_\ell b_{k-\ell}$
First difference	$x[n]-x[n-1] \;\leftrightarrow\; \left(1-e^{-jk(2\pi/N)}\right)a_k$
Running sum	$\sum_{r=-\infty}^{n} x[r] \;\leftrightarrow\; a_k/\left(1-e^{-jk(2\pi/N)}\right)$, only if $a_0 = 0$
Real signal	$a_{-k} = a_k^*$: $\lvert a_k\rvert$ even, $\angle a_k$ odd
Real and even	a_k real and even
Real and odd	a_k purely imaginary and odd
Even and odd parts	$\mathcal{E}\mathrm{v}\{x\} \leftrightarrow \mathrm{Re}\{a_k\}$, $\mathcal{O}\mathrm{dd}\{x\} \leftrightarrow j\,\mathrm{Im}\{a_k\}$, for real x
Parseval	$\frac{1}{N}\sum_{n=\langle N\rangle} \lvert x[n]\rvert^2 = \sum_{k=\langle N\rangle} \lvert a_k\rvert^2$

⚠ WHERE DISCRETE TIME DIFFERS

The coefficients repeat, $a_{k+N} = a_k$. So multiplication sums over one period, time scaling carries $1/m$, and a first difference takes the place of the derivative.

Checklist

1. Confirm that the signal is periodic and find T_0 or N before anything else.
2. Set $\omega_0 = 2\pi/T_0$ or $2\pi/N$ and write every component frequency as a multiple of it. That multiple is k.
3. If the signal is already a sum of sinusoids, read the coefficients off with Euler's relations. Otherwise integrate or sum, treating $k=0$ separately.
4. Compare a_0 with the average of the signal, and check the symmetry: a real signal gives $a_{-k} = a_k^*$.
5. Through an LTI system, $b_k = a_k H(jk\omega_0)$: one product per harmonic.
6. To return to a real signal, pair k with $-k$: amplitude $2\lvert b_k\rvert$, phase $\angle b_k$. The term b_0 stands alone, and in discrete time so does the term at $k = N/2$.

Where Chapter 5 begins

Everything in this chapter needed the signal to repeat. A single pulse does not repeat. It has no fundamental period and no harmonics to carry coefficients. The rectangular wave already showed the way out: its coefficients were samples of one envelope, taken every $\omega_0 = 2\pi/T_0$. Keep one pulse and let T_0 grow. The envelope stays the same while the samples crowd together and shrink as $1/T_0$.

WHERE CHAPTER 5 BEGINS

$$T_0 a_k = E(k\omega_0) \quad \xrightarrow{T_0\to\infty} \quad X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\,\mathrm{d}t$$

In the limit the stems merge into the curve they were always sampling. That curve is the Fourier transform, a continuous spectrum for a signal that does not repeat.

Exercises

4.1 Two periodic signals have fundamental periods $2/9$ s and $8/21$ s. Find the fundamental period of their sum, and check it by division.

Answer: $T_0 = \mathrm{LCM}(2,8)/\mathrm{GCD}(9,21) = 8/3$ s. Check: $(8/3)/(2/9) = 12$ and $(8/3)/(8/21) = 7$, both whole, and $\gcd(12,7) = 1$.

4.2 For $x(t) = 1 + \frac{1}{2}\cos(2\pi t) + \sin(3\pi t)$, state a_0 and explain in one sentence why it is not zero.

Answer: $a_0 = 1$: the constant term is the $k = 0$ harmonic, and it is also the average of the signal, since both sinusoids complete whole cycles in one period.

4.3 A rectangular wave has $T_0 = 8T_1$. Find a_0 and a_2.

Answer: $a_0 = 2T_1/T_0 = 1/4$ and $a_2 = \sin(\pi/2)/(2\pi) = 1/(2\pi) = 0.159$.

4.4 A real signal has $\omega_0 = 2$ rad/s, $a_2 = -1 + j$, $a_{-2} = a_2^*$, and every other coefficient zero. Write it in amplitude–phase form and in cosine–sine form.

Answer: $A_2 = \sqrt{2}$ and $\theta_2 = 3\pi/4$, since $-1 + j$ lies in the second quadrant. So $x(t) = 2\sqrt{2}\cos(4t + 3\pi/4)$. With $B_2 = -1$ and $C_2 = 1$, $x(t) = 2[-\cos 4t - \sin 4t] = -2\cos 4t - 2\sin 4t$.

4.5 Explain why the finite geometric sum used for the discrete-time square wave requires $r \neq 1$ and not $|r| < 1$, and say what happens at the excluded value.

Answer: A finite sum always has a value; only the closed form fails, where its denominator $1 - r$ vanishes. Here $|r| = 1$ at every k, so $|r| < 1$ would exclude the sum entirely. At $r = 1$, that is, k a multiple of N, every term is 1 and the sum is $2N_1 + 1$.

4.6 A filtered signal has $b_1 = 0.1517e^{-j1.263}$ and $\omega_0 = \pi$ rad/s. Write the contribution of the pair $k = \pm 1$ to the real output.

Answer: $2|b_1|\cos(\pi t + \angle b_1) = 0.303\cos(\pi t - 1.263)$.

4.7 A square wave and a triangular wave are truncated to the same number of harmonics. Which is approximated better, and which shows a Gibbs overshoot?

Answer: The triangular wave: its coefficients decay like $1/k^2$ against $1/k$, so its dropped power falls like $1/N^3$ rather than $1/N$. Only the square wave has a jump, so only it shows a Gibbs overshoot.

4.8 Find the gains at $\omega = 0$ and $\omega = \pi$ of the recursive filter $y[n] + 0.5\, y[n-1] = x[n]$. Is it low-pass or high-pass?

Answer: Here $a = -0.5$. $H(e^{j0}) = 1/(1-a) = 1/1.5 = \frac{2}{3}$ and $H(e^{j\pi}) = 1/(1+a) = 1/0.5 = 2$. The gain is largest at $\omega = \pi$, so the filter is high-pass.

CHAPTER

5

The continuous-time Fourier transform

The continuous-time Fourier transform describes a signal that does not repeat by a continuous function of frequency. To derive it, repeat a finite-duration signal with a period and let the period grow without bound. The harmonics crowd together, and the Fourier-series coefficients close onto one curve: the transform.

The transform is useful for the same reason as the Fourier series. Convolution in time becomes multiplication in frequency, for every signal that has a transform. The frequency response of an LTI system still says what the system does to each frequency. The chapter follows six steps: the transform itself (5.1), the standard pairs (5.2), periodic signals (5.3), the properties (5.4), convolution and multiplication (5.5), and systems described by a differential equation (5.6). Section 5.7 collects the results.

5.1 From series to transform

The periodic extension of a pulse

Let $x(t)$ be zero for $|t| > T_1$. The number T_1 is the half-width of the support and belongs to the signal alone. Build a periodic signal $\tilde{x}$ by repeating the pulse every T seconds, with $T > 2T_1$ so that the copies do not touch:

$$\tilde{x}(t) = \sum_{m=-\infty}^{\infty} x(t - mT), \qquad \tilde{x}(t) = \tilde{x}(t+T), \qquad T > 2T_1.$$

The extension is periodic, so the Fourier series of Chapter 4 applies to it. Inside one period $\tilde{x}$ equals the original pulse. The signal fixes T_1; only the period T is ours to choose. Increase T while keeping the central pulse fixed. The neighbouring copies move to larger values of $|t|$, and $\tilde{x}(t) \to x(t)$ for every t as $T \to \infty$.

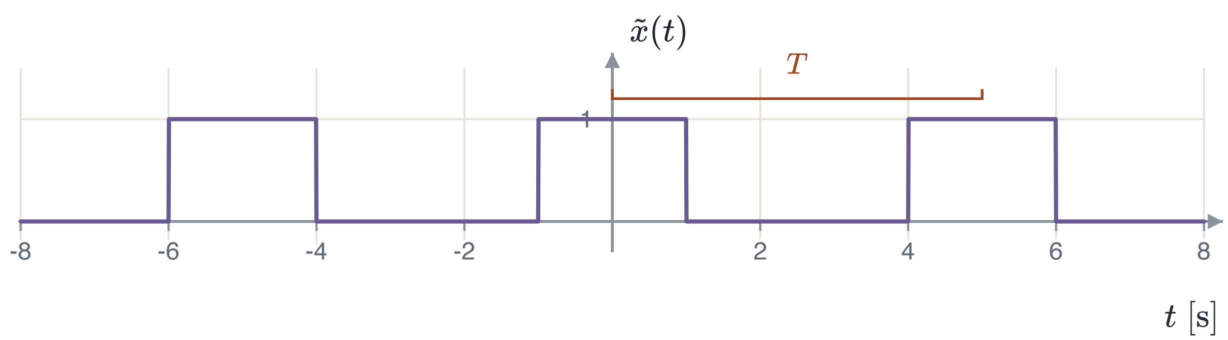

FIGURE 5.1 The periodic extension of the pulse $x(t) = 1$ on $|t| < 1$, with $T = 5T_1$. The condition $T > 2T_1$ keeps the copies apart.

The coefficients are samples of one curve

Apply the analysis equation of Chapter 4 to $\tilde{x}$ over one period. Inside that period $\tilde{x} = x$, and outside $|t| < T_1$ the integrand is zero, so the limits may be opened to all of time:

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)e^{-jk\omega_0 t}\,\mathrm{d}t = \frac{1}{T}\int_{-\infty}^{\infty} x(t)e^{-jk\omega_0 t}\,\mathrm{d}t, \qquad \omega_0 = \frac{2\pi}{T}.$$

The right-hand integral has the same form for every k; only the frequency $k\omega_0$ changes. Replace that frequency by the continuous variable ω and define the transform.

DEFINITION OF THE TRANSFORM

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\,e^{-j\omega t}\,\mathrm{d}t, \qquad T\,a_k = X(jk\omega_0)$$

$X(j\omega)$ is defined for every real ω, not only at the harmonics. Every coefficient of the periodic extension is one point of the curve X, scaled by $1/T$. The curve is built from the pulse alone, so a larger T cannot move it. A larger T only makes the spacing $\omega_0 = 2\pi/T$ smaller. For the pulse of half-width $T_1 = 1$ and the period $T = 8$, for example, $8a_0 = X(j0) = 2T_1 = 2$, and the same value holds at every period.

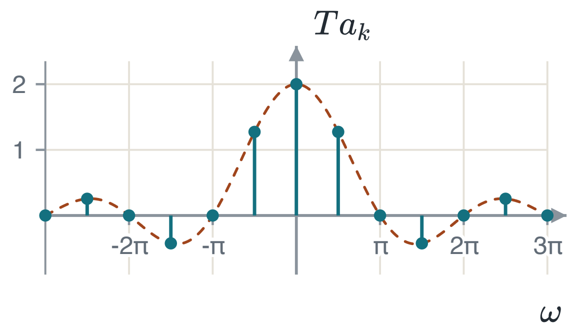

FIGURE 5.2 $T = 4T_1$.

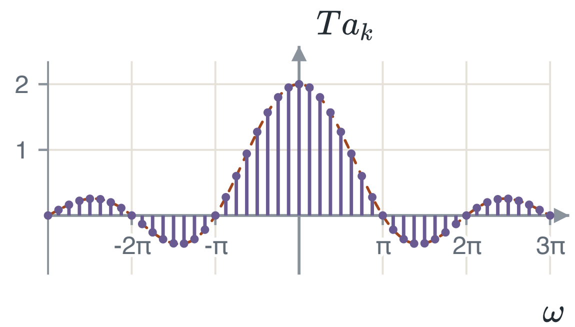

FIGURE 5.3 $T = 16T_1$: the same dashed curve, sampled four times as finely.

The sum becomes an integral

Put $a_k = \frac{1}{T}X(jk\omega_0)$ back into the synthesis equation of Chapter 4. Then replace $1/T$ by $\omega_0/2\pi$, which is the definition $\omega_0 = 2\pi/T$ solved for $1/T$. Nothing has been approximated yet:

$$\begin{aligned}\tilde{x}(t) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \\ &= \sum_{k=-\infty}^{\infty} \frac{1}{T} X(jk\omega_0)\,e^{jk\omega_0 t} \\ &= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(jk\omega_0)\,e^{jk\omega_0 t}\,\omega_0.\end{aligned}$$

Every term now contains the width ω_0, so the expression is a Riemann sum: a sum of strips of height $X(jk\omega_0)e^{jk\omega_0 t}$ and width ω_0. Let $T \to \infty$. Then $\tilde{x}(t) \to x(t)$, the width ω_0 becomes $\mathrm{d}\omega$, the sample frequencies cover the whole frequency axis, and the sum becomes an integral.

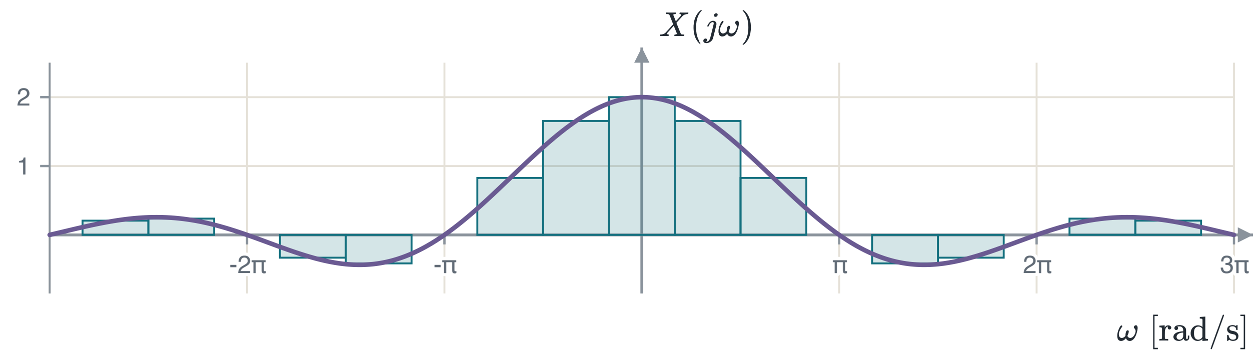

FIGURE 5.4 The strips of the sum for $T = 6T_1$, each of height $X(jk\omega_0)$ and width $\omega_0 = \pi/3$. As the width shrinks, the strips fill the area under the curve.

THE CONTINUOUS-TIME FOURIER TRANSFORM PAIR

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, \mathrm{d}t \qquad \text{(analysis)}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, \mathrm{d}\omega \qquad \text{(synthesis)}$$

Analysis integrates over t and leaves a function of ω: a signal goes in and a spectrum comes out. Synthesis integrates over ω and leaves a function of t: a spectrum goes in and a signal comes out. The factor $1/2\pi$ arrived as $\omega_0/2\pi$ when the spacing was substituted, and it stays on the synthesis side. Without it, every value of the rebuilt signal would be 2π times too large. The pair is written $x(t) \leftrightarrow X(j\omega)$, with the argument $j\omega$ and never ω alone.

⚠ WHICH EQUATION IS WHICH

The variable of integration settles it. Integrating time away produces a spectrum, and that is analysis. Integrating frequency away produces a signal, and that is synthesis. The analysis exponent carries the minus sign; the synthesis exponent is positive and carries the $1/2\pi$. The line $x(t) = \int X(j\omega)e^{-j\omega t}\,\mathrm{d}\omega$ therefore contains two errors: the factor $1/2\pi$ is missing, and the exponent must be $+j\omega t$.

The case of a letter carries meaning. The small letter names the signal and takes t. The capital letter names its spectrum and takes $j\omega$. In this course X is the spectrum of a signal, and H is the frequency response of a system.

When the transform exists

Two conditions each guarantee that $X(j\omega)$ exists. They are alternatives, joined by "or", and neither implies the other.

1. **Condition A, finite energy.** $\int_{-\infty}^{\infty} |x(t)|^2\, \mathrm{d}t < \infty$.
2. **Condition B, the Dirichlet conditions.** x is absolutely integrable, $\int_{-\infty}^{\infty} |x(t)|\, \mathrm{d}t < \infty$, and has finitely many maxima, minima and finite jumps in any finite interval.

Many signals meet both. The signal $e^{-2t}u(t)$ has area $\int_0^\infty e^{-2t}\,\mathrm{d}t = \left[e^{-2t}/(-2)\right]_0^\infty = 0 - (-\frac{1}{2}) = \frac{1}{2}$ and energy $\int_0^\infty e^{-4t}\,\mathrm{d}t = \left[e^{-4t}/(-4)\right]_0^\infty = \frac{1}{4}$; both are finite. That the conditions are independent is shown by one example each way. The signal $\sin(t)/t$ has finite energy, but its modulus decays only like

$1/|t|$, so its area diverges. The signal $1/\sqrt{t}$ on $0 < t < 1$ has area $\left[2\sqrt{t}\right]_0^1 = 2$, but its square is $1/t$, whose integral $\left[\ln t\right]_0^1$ diverges at the lower limit.

Convergence at a jump

The rectangular pulse $x(t) = 1$ for $|t| < 1$, and 0 elsewhere, meets the Dirichlet conditions. Its transform is $X(j\omega) = 2\sin\omega/\omega$ (Example 5.5 with $T_1 = 1$). Put this transform into the synthesis equation, but keep only the band $|\omega| < W$. The question is what the result $x_W(t)$ does as W grows, above all at the two jumps $t = \pm 1$.

$$x_W(t) = \frac{1}{2\pi}\int_{-W}^{W} \frac{2\sin\omega}{\omega}\, e^{j\omega t}\,\mathrm{d}\omega.$$

Write $e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$. The factor $2\sin\omega/\omega$ is even in ω. Its product with the odd function $\sin(\omega t)$ is odd, so that part integrates to zero over the symmetric band. Its product with $\cos(\omega t)$ is even, so that integral is twice the integral over $0 < \omega < W$. Then the identity $\sin\alpha\cos\beta = \frac{1}{2}\sin(\alpha+\beta) + \frac{1}{2}\sin(\alpha-\beta)$, with $\alpha = \omega$ and $\beta = \omega t$, splits the integrand in two:

$$\begin{aligned} x_W(t) &= \frac{1}{2\pi}\int_{-W}^{W} \frac{2\sin\omega}{\omega}\cos(\omega t)\,\mathrm{d}\omega \\ &= \frac{2}{\pi}\int_0^W \frac{\sin\omega\,\cos(\omega t)}{\omega}\,\mathrm{d}\omega \\ &= \frac{1}{\pi}\int_0^W \frac{\sin\big(\omega(1+t)\big) + \sin\big(\omega(1-t)\big)}{\omega}\,\mathrm{d}\omega. \end{aligned}$$

Each term becomes the sine integral $\mathrm{Si}(z) = \int_0^z \frac{\sin u}{u}\,\mathrm{d}u$ after one substitution. In the first term put $u = \omega(1+t)$, so $\mathrm{d}u/u = \mathrm{d}\omega/\omega$, and the limits $\omega = 0$ and $\omega = W$ become $u = 0$ and $u = W(1+t)$. The second term is the same with $1-t$ in place of $1+t$. The function Si is odd, so $\mathrm{Si}\big(W(1-t)\big) = -\mathrm{Si}\big(W(t-1)\big)$:

$$x_W(t) = \frac{1}{\pi}\Big[\mathrm{Si}\big(W(t+1)\big) - \mathrm{Si}\big(W(t-1)\big)\Big].$$

$\mathrm{Si}(z)$ rises from 0, overshoots, and settles at $\pi/2$ as $z \to \infty$. Its largest value is at $z = \pi$, where $\mathrm{Si}(\pi) = 1.851937$. Three facts follow.

1. **Away from the jumps, $x_W \to x$.** For $|t| < 1$, $W(t+1) \to +\infty$ and $W(t-1) \to -\infty$, so $x_W \to \frac{1}{\pi}\left[\frac{\pi}{2} + \frac{\pi}{2}\right] = 1$. For $|t| > 1$ both arguments have the same sign, and the two terms cancel in the limit: $x_W \to 0$.
2. **At a jump, x_W tends to the midpoint.** At $t = 1$, $x_W(1) = \frac{1}{\pi}\big[\mathrm{Si}(2W) - \mathrm{Si}(0)\big] = \frac{1}{\pi}\mathrm{Si}(2W) \to \frac{1}{\pi}\cdot\frac{\pi}{2} = \frac{1}{2}$.
3. **The overshoot does not shrink.** Just inside the jump, at $t = 1 - \pi/W$, the arguments are $W(t-1) = -\pi$ and $W(t+1) = 2W - \pi$. For large W the second is large, so $x_W \approx \frac{1}{\pi}\left[\frac{\pi}{2} + \mathrm{Si}(\pi)\right] = \frac{1}{2} + \frac{1.851937}{\pi} = 1.0895$. The peak moves towards the jump, a distance of about π/W away, but its height tends to 1.0895, not to 1.

The energy of the error still goes to zero. By Parseval's relation (Section 5.4) the error $x - x_W$ has energy $\frac{1}{2\pi}\int_{|\omega|>W} |X(j\omega)|^2\,\mathrm{d}\omega$, which is the tail of a finite integral and tends to 0. So the overshoot keeps its height while the energy of the error vanishes. The partial sums of a Fourier series behave the same way at a jump (Chapter 4).

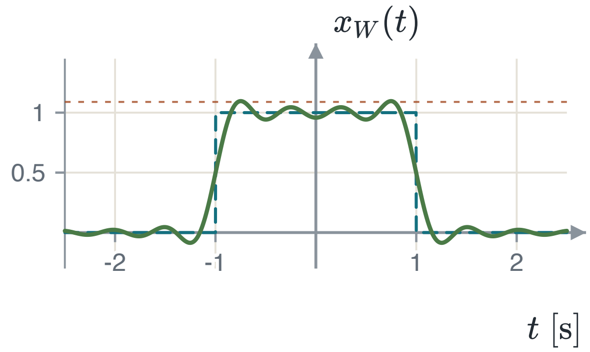

FIGURE 5.5 $W = 12$ rad/s. The highest value is 1.096.

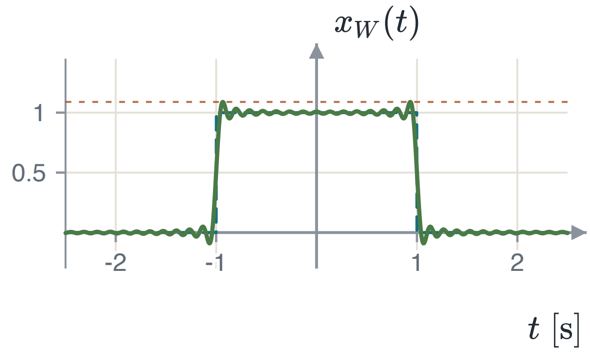

FIGURE 5.6 $W = 48$ rad/s. The ripples crowd to the jumps; the highest value is 1.089.

In both panels the dashed pulse is $x(t)$ and the fine dashed line marks 1.0895. At small W the ripples from the two jumps overlap and push the highest value a little above 1.0895.

Transforms in the limit

Neither condition is necessary. A constant, a complex exponential and every periodic signal have infinite energy and infinite area, so they fail both conditions. They still have spectra, in the limiting sense: the transform is an impulse, defined by what it does inside an integral, exactly as $\delta(t)$ is. Every property of this chapter still applies to such a transform.

Take the pair $e^{-a|t|} \leftrightarrow 2a/(a^2 + \omega^2)$ from Example 5.4 and let $a \to 0$. The signal tends to the constant 1. The transform grows tall at $\omega = 0$, where it equals $2a/a^2 = 2/a$, and narrow, since it falls to half that height at $\omega = \pm a$. For $a = 0.5$ the peak is already 4. Its area does not depend on a at all:

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{2a}{a^2 + \omega^2}\, \mathrm{d}\omega &= \left[2\tan^{-1}\frac{\omega}{a}\right]_{-\infty}^{\infty} \\ &= 2\left(\frac{\pi}{2}\right) - 2\left(-\frac{\pi}{2}\right) = 2\pi.\end{aligned}$$

A curve of fixed area 2π that becomes tall and narrow tends to an impulse of weight 2π. So $1 \leftrightarrow 2\pi\delta(\omega)$
.

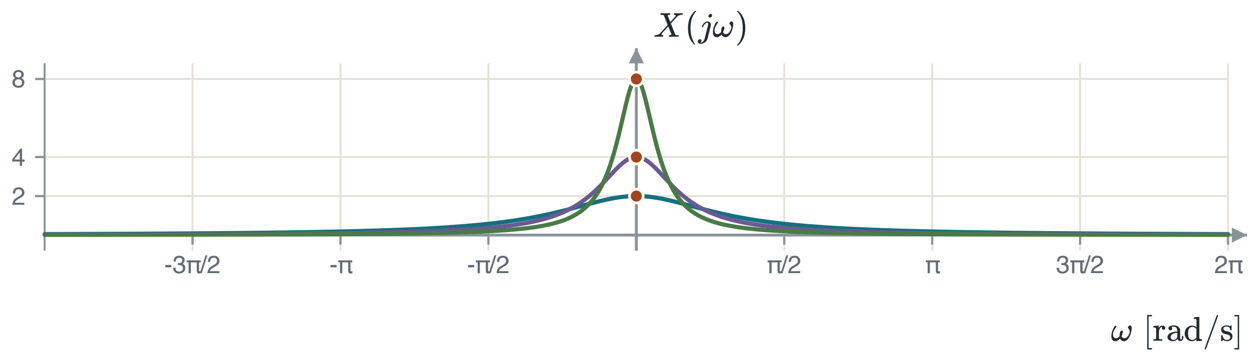

FIGURE 5.7 The transform $2a/(a^2 + \omega^2)$ for $a = 1$, 0.5 and 0.25, with peaks 2, 4 and 8. Every curve has area 2π.

5.2 The standard pairs

Most problems start from a few standard pairs. A property of Section 5.4 then turns a standard pair into the transform that is needed. This section derives the pairs.

EXAMPLE 5.1 — THE IMPULSE AND THE SHIFTED IMPULSE

GIVEN $x(t) = \delta(t)$, and then $x(t) = \delta(t - t_0)$ with t_0 a fixed time.

FIND $X(j\omega)$ in both cases, with magnitude and phase.

METHOD The signal is an impulse, so the sifting property evaluates the analysis integral directly. Substitute the impulse into the analysis equation.

SOLUTION Substitute $x(t) = \delta(t)$ into the analysis equation. The sifting property $\int \delta(t)g(t)\,\mathrm{d}t = g(0)$ evaluates the exponential at $t = 0$:

$$\begin{aligned}\mathcal{F}\{\delta(t)\} &= \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\,\mathrm{d}t \\ &= e^{-j\omega\cdot 0} = 1.\end{aligned}$$

Every frequency is present, equally, with no phase. For the shifted impulse the sifting happens at $t = t_0$, because $\delta(t - t_0)$ is zero everywhere except at that instant:

$$\begin{aligned}\mathcal{F}\{\delta(t - t_0)\} &= \int_{-\infty}^{\infty} \delta(t - t_0)\, e^{-j\omega t}\,\mathrm{d}t \\ &= e^{-j\omega t_0}.\end{aligned}$$

Write this in polar form: $e^{-j\omega t_0}$ has modulus 1 and angle $-\omega t_0$, so

$$|X(j\omega)| = 1, \qquad \angle X(j\omega) = -\omega t_0.$$

CHECK Push $e^{-j\omega t_0}$ back through the synthesis equation. Combine the two exponentials first:

$$\begin{aligned}x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega t_0}\, e^{j\omega t}\,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega(t-t_0)}\,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\cdot 2\pi\,\delta(t - t_0) = \delta(t - t_0).\end{aligned}$$

The last line uses $\int e^{j\omega\tau}\,\mathrm{d}\omega = 2\pi\delta(\tau)$, which is the pair $1 \leftrightarrow 2\pi\delta(\omega)$ of Section 5.1 with the roles of t and ω exchanged. Reading: moving a signal in time changes no magnitude; it turns each frequency component by an angle proportional to its frequency.

EXAMPLE 5.2 — ONE IMPULSE IN FREQUENCY

GIVEN $X(j\omega) = 2\pi\delta(\omega - \omega_0)$.

FIND $x(t)$.

METHOD The spectrum is an impulse, so use the synthesis equation and apply sifting in the variable ω .

SOLUTION Substitute the spectrum into the synthesis equation. The impulse sits at $\omega = \omega_0$, so sifting evaluates $e^{j\omega t}$ there:

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\,\delta(\omega-\omega_0)\,e^{j\omega t}\,\mathrm{d}\omega \\ &= \frac{2\pi}{2\pi}\int_{-\infty}^{\infty} \delta(\omega-\omega_0)\,e^{j\omega t}\,\mathrm{d}\omega \\ &= e^{j\omega_0 t}. \end{aligned}$$

The 2π of the impulse weight and the $1/2\pi$ of the synthesis equation cancel exactly. That is why the weight is written as 2π and not as 1.

CHECK $|x(t)| = 1$ and $\angle x(t) = \omega_0 t$. With an impulse of weight 1 the answer would be $\frac{1}{2\pi}e^{j\omega_0 t}$, which is not a unit-amplitude exponential. The factor sets the amplitude.

TWO CONSEQUENCES OF THE COMPLEX-EXPONENTIAL PAIR

$$e^{j\omega_0 t} \longleftrightarrow 2\pi\delta(\omega-\omega_0), \qquad 1 \longleftrightarrow 2\pi\delta(\omega)$$

The second is the first at $\omega_0 = 0$, and it agrees with the limit taken in Section 5.1.

EXAMPLE 5.3 – THE ONE-SIDED EXPONENTIAL

GIVEN $x(t) = e^{-at}u(t)$, with a a real constant.

FIND $X(j\omega)$, the condition on a, and the magnitude and phase.

METHOD The unit step makes the signal zero for negative time. Apply the analysis equation from 0 to ∞, and evaluate the antiderivative at both limits.

SOLUTION Substitute the signal into the analysis equation. The step $u(t)$ is zero for $t < 0$ and 1 for $t > 0$, so the lower limit becomes 0 and the step disappears. Then combine the two exponentials into one:

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} e^{-at}u(t)\,e^{-j\omega t}\,\mathrm{d}t \\ &= \int_{0}^{\infty} e^{-at}e^{-j\omega t}\,\mathrm{d}t \\ &= \int_{0}^{\infty} e^{-(a+j\omega)t}\,\mathrm{d}t. \end{aligned}$$

The integrand is an exponential in t with constant coefficient $-(a+j\omega)$, so its antiderivative is the same exponential divided by that coefficient:

$$X(j\omega) = \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_0^\infty$$
$$= \frac{1}{-(a+j\omega)}\left[\lim_{t\to\infty} e^{-(a+j\omega)t} - e^0\right]$$
$$= \frac{1}{-(a+j\omega)}\left[0-1\right] = \frac{1}{a+j\omega}.$$

The limit at the upper end is zero only when $a > 0$: $\left|e^{-(a+j\omega)t}\right| = e^{-at}\left|e^{-j\omega t}\right| = e^{-at}$, because $\left|e^{-j\omega t}\right| = 1$, and $e^{-at} \to 0$ needs $a > 0$. For $a \le 0$ the integral does not converge and the transform does not exist. So

$$e^{-at}u(t) \;\longleftrightarrow\; \frac{1}{a+j\omega}, \qquad a > 0.$$

For the magnitude, the modulus of a quotient is the quotient of the moduli, and $|a+j\omega| = \sqrt{a^2+\omega^2}$:

$$|X(j\omega)| = \frac{|1|}{|a+j\omega|} = \frac{1}{\sqrt{a^2+\omega^2}}.$$

For the phase, the angle of a quotient is the angle of the numerator minus the angle of the denominator. The number $a+j\omega$ has real part $a > 0$ and imaginary part ω, so its angle is $\tan^{-1}(\omega/a)$:

$$\angle X(j\omega) = \angle 1 - \angle(a+j\omega) = 0 - \tan^{-1}(\omega/a) = -\tan^{-1}(\omega/a).$$

CHECK

The magnitude is largest at $\omega = 0$, where $|X(j0)| = 1/a$: 5, 1 and 0.2 for $a = 0.2$, 1 and 5. A small a decays slowly in time and gives a tall, narrow spectrum; a large a decays fast and gives a low, wide one. For $a = 2$, $|X(j2)| = 1/\sqrt{4+4} = 1/(2\sqrt{2}) = 0.353553$. At $a = 1$ the phase is $-\tan^{-1}1 = -\pi/4 = -0.785398$ rad at $\omega = 1$, and $-\tan^{-1}\sqrt{3} = -\pi/3 = -1.047198$ rad at $\omega = \sqrt{3}$. Both are negative, and the phase curve falls from $+\pi/2$ to $-\pi/2$. A curve that rises belongs to $+\tan^{-1}(\omega/a)$, which is the angle of $a+j\omega$, the reciprocal of the answer.

⚠ WHERE THAT MINUS SIGN IS LOST

The numerator is the real number 1, so its angle is zero, and it is tempting to skip that term and copy $\tan^{-1}(\omega/a)$ straight out of the denominator. What is then reported is $\angle(a+j\omega)$, the angle of the reciprocal of the answer. Write the subtraction out, including the term that is zero.

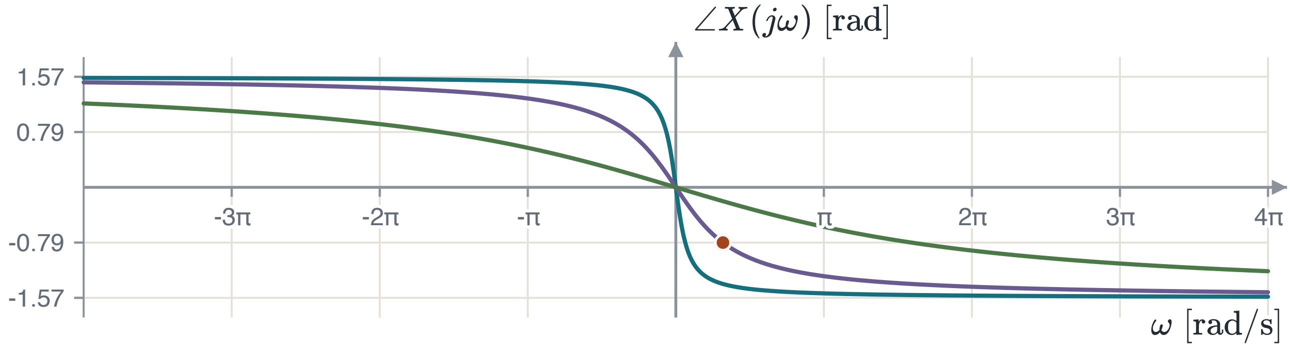

FIGURE 5.8 The phase of $1/(a+j\omega)$ for $a = 0.2$ (steepest), 1 and 5 (flattest). The marked value is $-\pi/4$ at $a = \omega = 1$.

EXAMPLE 5.4 — THE TWO-SIDED EXPONENTIAL

GIVEN $x(t) = e^{-a|t|}$ with $a > 0$.

FIND $X(j\omega)$.

METHOD The absolute value gives different exponential formulas for negative and positive time. Split the analysis integral at $t = 0$; on the negative-time interval $|t| = -t$ and the exponent is $+at$.

SOLUTION For $t < 0$ the signal is $e^{-a(-t)} = e^{at}$, and for $t > 0$ it is e^{-at}. Split the analysis integral at $t = 0$ and combine the exponentials in each part:

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{0} e^{at}e^{-j\omega t}\,\mathrm{d}t + \int_{0}^{\infty} e^{-at}e^{-j\omega t}\,\mathrm{d}t \\ &= \int_{-\infty}^{0} e^{(a-j\omega)t}\,\mathrm{d}t + \int_{0}^{\infty} e^{-(a+j\omega)t}\,\mathrm{d}t. \end{aligned}$$

The second integral is Example 5.3 and equals $1/(a+j\omega)$. Evaluate the first with its antiderivative. At the lower end, $|e^{(a-j\omega)t}| = e^{at} \to 0$ as $t \to -\infty$ because $a > 0$:

$$\begin{aligned} \int_{-\infty}^{0} e^{(a-j\omega)t}\,\mathrm{d}t &= \left[\frac{e^{(a-j\omega)t}}{a-j\omega}\right]_{-\infty}^{0} \\ &= \frac{e^0 - 0}{a-j\omega} = \frac{1}{a-j\omega}. \end{aligned}$$

Add the two fractions over the common denominator $(a-j\omega)(a+j\omega) = a^2 - (j\omega)^2 = a^2 + \omega^2$:

$$\begin{aligned} X(j\omega) &= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} \\ &= \frac{(a+j\omega)+(a-j\omega)}{(a-j\omega)(a+j\omega)} \\ &= \frac{2a}{a^2+\omega^2}, \qquad a > 0. \end{aligned}$$

The imaginary parts $+j\omega$ and $-j\omega$ cancelled in the numerator, so the transform is real.

CHECK $X(j0) = 2a/a^2 = 2/a$ gives 4, 2 and 0.4 for $a = 0.5$, 1 and 5. The signal is real and even, and the transform came out real and even, which is the general rule proved in Section 5.4. The transform is never zero: at $a = 1$ and $\omega = 10^6$ it is still $2/(1 + 10^{12}) \approx 2 \times 10^{-12}$.

EXAMPLE 5.5 — THE RECTANGULAR PULSE

GIVEN $x(t) = 1$ for $|t| < T_1$ and 0 otherwise.

FIND $X(j\omega)$, its value at the origin, and its zeros.

METHOD The signal is non-zero only on $-T_1 < t < T_1$, so restrict the analysis integral to that support and integrate $e^{-j\omega t}$. Then use $e^{j\theta} - e^{-j\theta} = 2j\sin\theta$.

SOLUTION Substitute the pulse into the analysis equation. Outside $|t| < T_1$ the signal is zero, so those parts of the integral contribute nothing, and inside the signal equals 1:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\,\mathrm{d}t = \int_{-T_1}^{T_1} 1 \cdot e^{-j\omega t}\,\mathrm{d}t.$$

For $\omega \neq 0$ the antiderivative of $e^{-j\omega t}$ is $e^{-j\omega t}/(-j\omega)$. Evaluate at the two limits, then move the minus sign of the denominator into the numerator by swapping the two terms:

$$\begin{aligned} X(j\omega) &= \left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-T_1}^{T_1} \\ &= \frac{e^{-j\omega T_1} - e^{j\omega T_1}}{-j\omega} \\ &= \frac{e^{j\omega T_1} - e^{-j\omega T_1}}{j\omega}. \end{aligned}$$

Euler gives $e^{j\theta} - e^{-j\theta} = (\cos\theta + j\sin\theta) - (\cos\theta - j\sin\theta) = 2j\sin\theta$. With $\theta = \omega T_1$:

$$X(j\omega) = \frac{2j\sin(\omega T_1)}{j\omega} = \frac{2\sin(\omega T_1)}{\omega} = 2T_1\,\mathrm{sinc}(\omega T_1),$$

with the unnormalised sinc, $\mathrm{sinc}(\theta) = \sin\theta/\theta$. The j cancels, so the answer is real. At $\omega = 0$ the integrand is 1 and the integral is the width $2T_1$ directly.

CHECK At the origin the formula reads $0/0$. L'Hôpital, differentiating numerator and denominator with respect to ω, gives $\lim_{\omega\to 0} 2T_1\cos(\omega T_1)/1 = 2T_1$. This agrees with the direct value and is the area under the pulse: 2, 10 and 20 for $T_1 = 1$, 5 and 10. The zeros are where $\sin(\omega T_1) = 0$ with $\omega \neq 0$, that is $\omega T_1 = \pm k\pi$, so

$$\omega = \pm\frac{k\pi}{T_1}, \qquad k = 1, 2, 3, \ldots$$

The origin is excluded, because there the limit is the peak, not a zero. Writing $k \in \mathbb{Z}$ would call the peak a zero.

THE SINC CONVENTION USED THROUGHOUT

This course writes $\operatorname{sinc}(\theta) = \dfrac{\sin\theta}{\theta}$, the **unnormalised** sinc, with $\operatorname{sinc}(0) = 1$ and zeros at $\theta = \pm\pi, \pm 2\pi, \ldots$ In that convention the pulse pair reads $X(j\omega) = 2T_1 \operatorname{sinc}(\omega T_1)$. Signal-processing software and many texts use the **normalised** sinc, $\sin(\pi\theta)/(\pi\theta)$, whose first positive zero is at $\theta = 1$ and whose zeros are the non-zero integers. There the same pair is written with the argument $\omega T_1/\pi$. The two expressions are equal, but the arguments differ by a factor of π. An argument copied between the conventions agrees at the origin and is wrong everywhere else, so the convention is stated at every point of use. The peak $X(j0) = 2T_1$ is the same in either convention.

The part of the spectrum on $|\omega| < \pi/T_1$ is the **main lobe**; its width is $2\pi/T_1$. Beyond it the **side lobes** alternate in sign and shrink like $1/|\omega|$, because $|2\sin(\omega T_1)/\omega| \le 2/|\omega|$. For $T_1 = 1$ the first side lobe has its extreme where the derivative of $\sin\omega/\omega$ is zero, that is where $\tan\omega = \omega$, at $\omega = 4.493409$; there $X = 2\sin(4.493409)/4.493409 = -0.434467$.

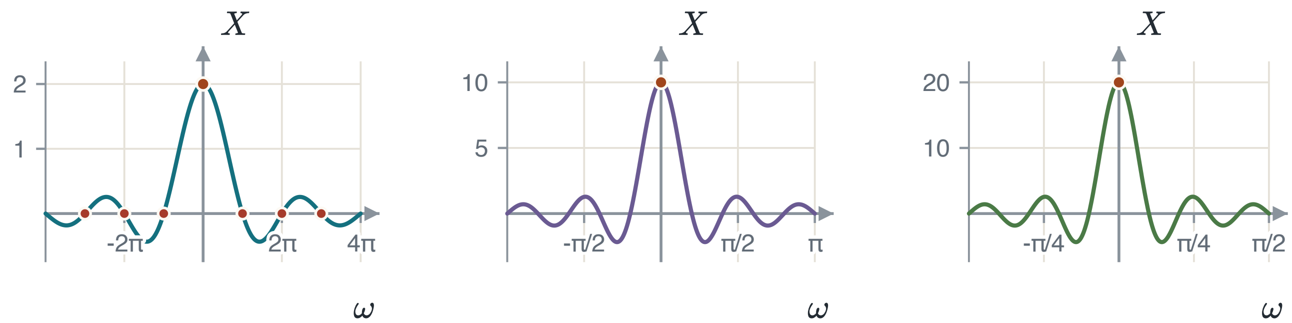

FIGURE 5.9 $T_1 = 1$: peak 2, zeros at $\pm\pi, \pm 2\pi, \ldots$

FIGURE 5.10 $T_1 = 5$: peak 10.

FIGURE 5.11 $T_1 = 10$: peak 20.

EXAMPLE 5.6 — THE IDEAL LOW-PASS BAND

GIVEN $X(j\omega) = 1$ for $|\omega| < W$ and 0 otherwise: the **ideal low-pass band**, with band edge W in rad/s.

FIND $x(t)$, its peak, and its zeros.

METHOD The spectrum is given and is non-zero only on $-W < \omega < W$, so apply the synthesis equation over those limits and keep the factor $1/2\pi$.

SOLUTION Substitute the spectrum into the synthesis equation. Outside $|\omega| < W$ it is zero, and inside it equals 1:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, \mathrm{d}\omega = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}\, \mathrm{d}\omega.$$

Here t is a fixed parameter and ω is the variable, so for $t \neq 0$ the antiderivative of $e^{j\omega t}$ is $e^{j\omega t}/(jt)$:

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\left[\frac{e^{j\omega t}}{jt}\right]_{-W}^{W} \\ &= \frac{e^{jWt} - e^{-jWt}}{2\pi jt} \\ &= \frac{2j\sin(Wt)}{2\pi jt} \\ &= \frac{\sin(Wt)}{\pi t}. \end{aligned}$$

The third line is Euler again, $e^{j\theta} - e^{-j\theta} = 2j\sin\theta$ with $\theta = Wt$; the $2j$ then cancels against the $2\pi j$. With the unnormalised sinc, $\operatorname{sinc}(\theta) = \sin\theta/\theta$, multiply and divide by W:

$$x(t) = \frac{W}{\pi}\cdot\frac{\sin(Wt)}{Wt} = \frac{W}{\pi}\operatorname{sinc}(Wt).$$

CHECK

At $t = 0$ the synthesis integral is $\frac{1}{2\pi}\int_{-W}^{W} 1\,\mathrm{d}\omega = \frac{2W}{2\pi} = \frac{W}{\pi}$. L'Hôpital on $\sin(Wt)/(\pi t)$ gives $\lim_{t\to 0} W\cos(Wt)/\pi = W/\pi$ as well: 0.5, 1 and 2 for $W = 0.5\pi$, π and 2π. The zeros are where $\sin(Wt) = 0$ with $t \neq 0$: $t = \pm k\pi/W$, $k = 1, 2, \ldots$, again with the origin excluded. This signal is not a pulse: it rings on both sides of the origin for ever, alternating in sign.

Examples 5.5 and 5.6 are one statement seen twice: a rectangle in either domain is a sinc in the other. Section 5.4 gives that symmetry a name, duality, and a proof.

Narrow in time, wide in frequency

Name the measures first. Take the full duration $T = 2T_1$ of the pulse and its **first-null bandwidth** $\mathrm{BW} = \pi/T_1$ rad/s, the distance from the origin to the first zero. Halve T_1 and the first zero doubles. The product does not depend on the width:

$$T \times \mathrm{BW} = 2T_1 \cdot \frac{\pi}{T_1} = 2\pi \qquad \text{at every width.}$$

For $T_1 = 1$ the product is $2 \times \pi$; for $T_1 = 1/4$ it is $0.5 \times 4\pi$. The pulse made four times narrower has a spectrum four times wider and four times lower.

⚠ THE VALUE DEPENDS ON THE WAVEFORM

The product is fixed for scaled versions of one waveform, because time scaling changes duration and frequency width by reciprocal factors (Section 5.4). It is not one constant for all shapes. A triangular pulse of the same total duration $2T_1$ has the transform $T_1\operatorname{sinc}^2(\omega T_1/2)$, with $\operatorname{sinc}(\theta) = \sin\theta/\theta$, because that triangle is the convolution of two rectangles of half-width $T_1/2$ and convolution multiplies transforms (Section 5.5). Its first null is where $\omega T_1/2 = \pi$, that is at $\omega = 2\pi/T_1$, so its product is $2T_1 \cdot 2\pi/T_1 = 4\pi$. In general, narrowing a signal in time widens its spectrum; both widths cannot be made arbitrarily small.

Time limitation and band limitation

A signal is **band-limited** when $X(j\omega) = 0$ for every $|\omega| > W$, for some finite W. Small is not enough; the spectrum must be exactly zero. The signal $\sin(3t)/(\pi t)$ is band-limited: by Example 5.6 with $W = 3$, its transform is 1 for $|\omega| < 3$ and 0 beyond.

$$\text{finite duration} \implies \text{not band-limited.}$$

This implication is a theorem. Read backwards, it says that a band-limited signal cannot have finite duration. The pulse of Example 5.5 illustrates it: its duration is finite, and its sinc transform is non-zero at arbitrarily high frequencies. The converse is false: infinite duration guarantees nothing. The signal $e^{-|t|}$ lasts for ever, and its transform $2/(1+\omega^2)$ is strictly positive at every finite frequency; at $\omega = 10^6$ it is about 2×10^{-12}, small but not zero. Chapter 7 depends on this difference.

5.3 Periodic signals

A periodic signal has a Fourier series, and the series can be transformed term by term. Linearity moves the transform inside the sum and past each constant a_k. Each term is then a complex exponential at frequency $k\omega_0$, and Example 5.2 gives its transform, $e^{jk\omega_0 t} \leftrightarrow 2\pi\delta(\omega - k\omega_0)$:

$$\begin{aligned} X(j\omega) &= \mathcal{F}\Big\{ \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \Big\} \\ &= \sum_{k=-\infty}^{\infty} a_k\, \mathcal{F}\{e^{jk\omega_0 t}\} \\ &= \sum_{k=-\infty}^{\infty} a_k \cdot 2\pi\, \delta(\omega - k\omega_0). \end{aligned}$$

TRANSFORM OF A PERIODIC SIGNAL

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \longleftrightarrow X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k\, \delta(\omega - k\omega_0), \qquad \omega_0 = \frac{2\pi}{T_0}$$

The spectrum is a train of impulses at the harmonics $k\omega_0$, and the impulse at $k\omega_0$ carries weight $2\pi a_k$. The coefficient and the weight are different objects: a_k multiplies a unit exponential, while $2\pi a_k$ is the area of an impulse in a spectrum. Reporting the coefficients as the transform loses a factor of 2π at every harmonic. A constant $x(t) = 3$, for example, has $a_0 = 3$ and an impulse of weight $2\pi a_0 = 6\pi$ at $\omega = 0$.

EXAMPLE 5.7 — THE LINE SPECTRUM OF A SQUARE WAVE

GIVEN The rectangular wave of Chapter 4: value 1 on $|t| < T_1$ in each period T, with $T_1 = 1$, for $T = 8T_1$, $16T_1$ and $32T_1$.

FIND The impulse weights, the spacing, and the weight at $\omega = 0$ for each period.

METHOD Take the coefficients a_k of the rectangular wave from Chapter 4. Multiply each by 2π and place an impulse at $k\omega_0$.

SOLUTION The coefficients are $a_0 = 2T_1/T$ and $a_k = \sin(k\omega_0 T_1)/(k\pi)$ for $k \neq 0$. Multiply by 2π; the π in a_k cancels against the 2π:

$$2\pi a_k = 2\pi \cdot \frac{\sin(k\omega_0 T_1)}{k\pi} = \frac{2\sin(k\omega_0 T_1)}{k},$$
$$2\pi a_0 = 2\pi \cdot \frac{2T_1}{T} = \frac{4\pi T_1}{T}.$$

With $T_1 = 1$, the spacing is $\omega_0 = 2\pi/T$ and the weight at the origin is $4\pi/T$:

$$\begin{aligned} T = 8T_1: &\quad \omega_0 = \pi/4 = 0.785398, \quad 2\pi a_0 = \pi/2 = 1.5708,\\ T = 16T_1: &\quad \omega_0 = \pi/8 = 0.392699, \quad 2\pi a_0 = \pi/4 = 0.7854,\\ T = 32T_1: &\quad \omega_0 = \pi/16 = 0.196350, \quad 2\pi a_0 = \pi/8 = 0.3927. \end{aligned}$$

Each doubling of the period halves the spacing and halves every weight.

CHECK For $T = 8T_1$ the weights are $2\sin(k\pi/4)/k$. At $k = \pm 4$ and ± 8 the sine is $\sin(\pm\pi) = 0$ or $\sin(\pm 2\pi) = 0$, so those impulses vanish. The weights are even in k. For $k = 5, 6, 7$ the angle $k\pi/4$ lies between π and 2π, so the sine is negative, and the impulses at $k = \pm 5, \pm 6, \pm 7$ point down. A plot of $|a_k|$ would hide both facts. The weights sit on the envelope $\omega_0 \cdot 2\sin(\omega T_1)/\omega$, whose shape does not change with T: this is the derivation of Section 5.1, carried out three times.

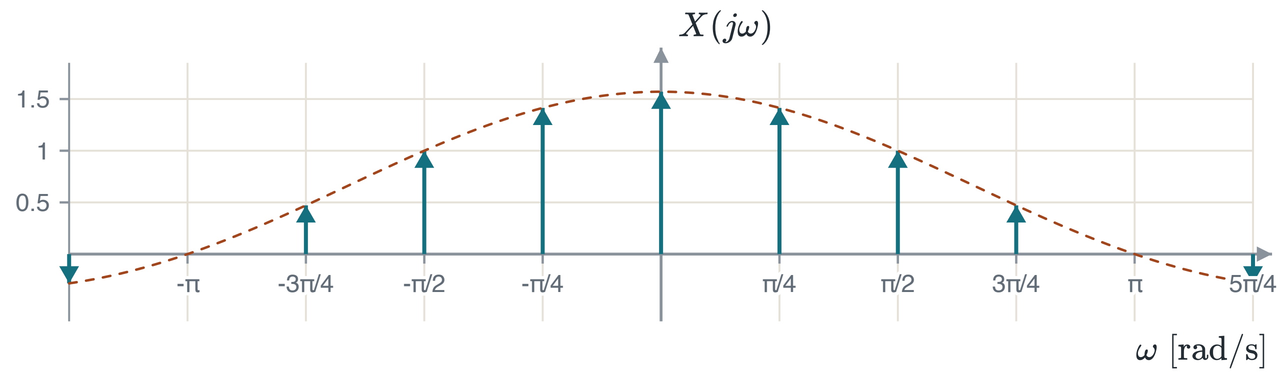

FIGURE 5.12 The square wave with $T = 8T_1$: impulses of weight $2\pi a_k$ at $k\pi/4$ sample the dashed envelope. The impulse at $\pm\pi$ is missing and those at $\pm 5\pi/4$ point down.

EXAMPLE 5.8 — THE LINE SPECTRUM OF A COSINE

GIVEN $x(t) = 4\cos(3\pi t)$.

FIND $X(j\omega)$.

METHOD A cosine is a sum of two complex exponentials. Expand it with Euler's relation and transform each exponential with $e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)$.

SOLUTION Euler's relation $\cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$ with $\theta = 3\pi t$ gives

$$4\cos(3\pi t) = 4 \cdot \tfrac{1}{2}\left(e^{j3\pi t} + e^{-j3\pi t}\right) = 2e^{j3\pi t} + 2e^{-j3\pi t}.$$

Transform each exponential and multiply by its coefficient:

$$\begin{aligned} \mathcal{F}\{4\cos(3\pi t)\} &= 2 \cdot 2\pi\,\delta(\omega - 3\pi) + 2 \cdot 2\pi\,\delta(\omega + 3\pi)\\ &= 4\pi\,\delta(\omega - 3\pi) + 4\pi\,\delta(\omega + 3\pi). \end{aligned}$$

CHECK There are two impulses, one at $+3\pi$ and one at -3π, each of weight $4\pi = 12.5664$. In the language of the Fourier series, $a_{\pm 1} = 2$ and each weight is $2\pi a_{\pm 1} = 4\pi$. For a cosine of unit amplitude the same steps give $\cos(\omega_0 t) \leftrightarrow \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$.

EXAMPLE 5.9 — THE LINE SPECTRUM OF A SINE

GIVEN $x(t) = 6\sin(4\pi t)$.

FIND $X(j\omega)$, and whether its weights are real or imaginary.

METHOD As in Example 5.8, with Euler's relation for the sine, $\sin\theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$.

SOLUTION With $\theta = 4\pi t$:

$$6\sin(4\pi t) = \frac{6}{2j}\,e^{j4\pi t} - \frac{6}{2j}\,e^{-j4\pi t} = \frac{3}{j}\,e^{j4\pi t} - \frac{3}{j}\,e^{-j4\pi t}.$$

Transform each exponential and multiply by its coefficient:

$$\begin{aligned}\mathcal{F}\{6\sin(4\pi t)\} &= \frac{3}{j}\cdot 2\pi\,\delta(\omega - 4\pi) - \frac{3}{j}\cdot 2\pi\,\delta(\omega + 4\pi)\\ &= \frac{6\pi}{j}\,\delta(\omega - 4\pi) - \frac{6\pi}{j}\,\delta(\omega + 4\pi).\end{aligned}$$

Since $1/j = -j$, the weights are $-j6\pi$ at $+4\pi$ and $+j6\pi$ at -4π: imaginary, of modulus $6\pi = 18.8496$, with opposite signs on the two sides.

CHECK The signal is real, so its transform must satisfy $X(-j\omega) = X^*(j\omega)$ (Section 5.4). The weight at -4π is $j6\pi$, and the conjugate of the weight at $+4\pi$ is $(-j6\pi)^* = j6\pi$. They agree. A spectrum with only the impulse at $+4\pi$ would belong to a complex signal.

EXAMPLE 5.10 — A CONSTANT, A COSINE AND A SINE

GIVEN $x(t) = 5 + 4\cos(3\pi t) + 6\sin(4\pi t)$.

FIND $X(j\omega)$, drawn so that both size and phase can be read.

METHOD Linearity: transform the three terms separately and add. The constant is the case $\omega_0 = 0$ of Example 5.2; the other two terms are Examples 5.8 and 5.9.

SOLUTION The constant gives $5\cdot 2\pi\,\delta(\omega) = 10\pi\,\delta(\omega)$. Add the results of Examples 5.8 and 5.9:

$$\begin{aligned}X(j\omega) = 10\pi\delta(\omega) &+ 4\pi\delta(\omega - 3\pi) + 4\pi\delta(\omega + 3\pi)\\ &+ \frac{6\pi}{j}\delta(\omega - 4\pi) - \frac{6\pi}{j}\delta(\omega + 4\pi).\end{aligned}$$

CHECK The signal is real, so the magnitude must be even in ω and the phase odd. Magnitudes: $10\pi = 31.4159$ at 0, $4\pi = 12.5664$ at $\pm 3\pi$, and $6\pi = 18.8496$ at $\pm 4\pi$. Phases: 0 except at $\pm 4\pi$, where they are $\mp\pi/2$ because $1/j = -j$. Both conditions hold.

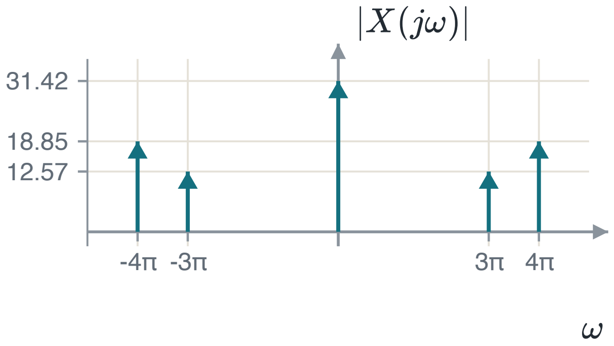

FIGURE 5.13 The magnitude, even in ω. It cannot tell the cosine from the sine.

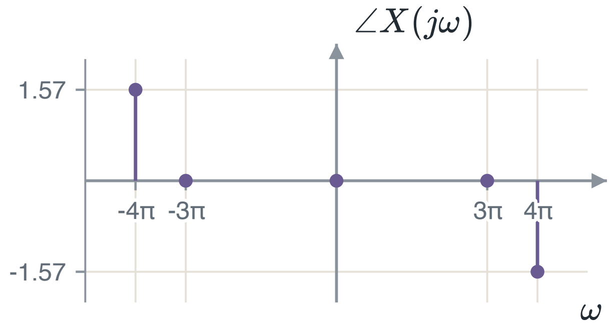

FIGURE 5.14 The phase in rad, odd in ω. It separates the sine from the cosine.

> ⚠ DRAWING A COMPLEX SPECTRUM
>
> Three weights are real and two are imaginary. One vertical axis cannot represent both signed real values and signed imaginary values. Plot magnitude and phase, or plot real part and imaginary part, and name the chosen pair in the caption.

EXAMPLE 5.11 — THE IMPULSE TRAIN

GIVEN $x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$, one unit impulse every T seconds.

FIND $X(j\omega)$.

METHOD The signal is periodic, so its transform is an impulse at each harmonic. Find its Fourier-series coefficients a_k over a period that holds exactly one impulse, then give each impulse the weight $2\pi a_k$.

SOLUTION The signal repeats every T, so $\omega_0 = 2\pi/T$. Apply the analysis equation of the Fourier series over the period $-T/2 < t < T/2$. That interval encloses exactly one impulse, the one at $t = 0$, so inside it the signal is just $\delta(t)$, and sifting evaluates the exponential at $t = 0$:

$$
\begin{aligned}
a_k &= \frac{1}{T}\int_{-T/2}^{T/2} \sum_m \delta(t - mT)\, e^{-jk\omega_0 t}\, \mathrm{d}t \\
&= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{-jk\omega_0 t}\, \mathrm{d}t \\
&= \frac{1}{T} e^0 = \frac{1}{T} \qquad \text{for every } k.
\end{aligned}
$$

Now use the transform of a periodic signal: each harmonic $k\omega_0 = 2\pi k/T$ carries an impulse of weight $2\pi a_k = 2\pi/T$:

$$
\begin{aligned}
X(j\omega) &= \sum_k 2\pi a_k\, \delta(\omega - k\omega_0) \\
&= \sum_k \frac{2\pi}{T}\, \delta\!\left(\omega - \frac{2\pi k}{T}\right).
\end{aligned}
$$

Hence

$$\sum_k \delta(t-kT) \longleftrightarrow \frac{2\pi}{T}\sum_k \delta\left(\omega-\frac{2\pi k}{T}\right).$$

CHECK Spacing and weight are the same number, $2\pi/T$: 6.2832 for $T=1$ and 3.1416 for $T=2$. Spreading the impulses twice as far apart in time brings them twice as close in frequency and makes them half as tall. This one pair is the mechanism behind sampling, in Chapter 7. The limits must enclose exactly one impulse; writing both as $-T/2$ encloses none.

5.4 Properties

Each property below says what one operation on a signal does to its transform. Each one is proved from the analysis or the synthesis equation. With the properties, one standard pair serves a whole family of signals. Section 5.7 collects them in one table.

Linearity and time shift

The analysis equation is an integral, and integration is linear, so $a\,x_1(t)+b\,x_2(t) \leftrightarrow a\,X_1(j\omega)+b\,X_2(j\omega)$ for any constants a and b.

For the time shift, put $x(t-t_0)$ into the analysis equation and substitute $\tau=t-t_0$, so $t=\tau+t_0$ and $\mathrm{d}t=\mathrm{d}\tau$. The limits are infinite and do not move, because $\tau\to\pm\infty$ exactly when $t\to\pm\infty$. Then split the exponential and take the factor that does not depend on τ outside:

$$\begin{aligned}\mathcal{F}\{x(t-t_0)\} &= \int_{-\infty}^{\infty} x(t-t_0)\,e^{-j\omega t}\,\mathrm{d}t\\ &= \int_{-\infty}^{\infty} x(\tau)\,e^{-j\omega(\tau+t_0)}\,\mathrm{d}\tau\\ &= e^{-j\omega t_0}\int_{-\infty}^{\infty} x(\tau)\,e^{-j\omega\tau}\,\mathrm{d}\tau\\ &= e^{-j\omega t_0}X(j\omega).\end{aligned}$$

The last integral is the analysis equation with τ as the dummy variable, so it is $X(j\omega)$. Since $|e^{-j\omega t_0}|=1$, no magnitude changes at any frequency. A delay of t_0 seconds adds the linear phase $-\omega t_0$: a delay of 2 s changes the phase at $\omega=1$ rad/s by -2 rad.

EXAMPLE 5.12 — A SUM OF SHIFTED PULSES

GIVEN $x_1(t)=1$ on $|t|<2$ and $x_2(t)=1$ on $|t|<1$, both zero elsewhere, and $x_3(t)=2x_1(t-4)+x_2(t-3)$.

FIND $X_3(j\omega)$ and $X_3(j0)$.

METHOD Transform each pulse with Example 5.5. Apply the time shift to each, then add with linearity.

SOLUTION Example 5.5 with $T_1=2$ and $T_1=1$ gives

$$X_1(j\omega)=\frac{2\sin(2\omega)}{\omega},\qquad X_2(j\omega)=\frac{2\sin\omega}{\omega}.$$

The time shift gives $x_1(t-4) \leftrightarrow e^{-j4\omega}X_1(j\omega)$ and $x_2(t-3) \leftrightarrow e^{-j3\omega}X_2(j\omega)$. Linearity adds them with their coefficients:

$$X_3(j\omega) = 2e^{-j4\omega}\,\frac{2\sin(2\omega)}{\omega} + e^{-j3\omega}\,\frac{2\sin\omega}{\omega}.$$

CHECK At $\omega = 0$ the analysis equation reads $X(j0) = \int x(t)\,\mathrm{d}t$, the area. Here $e^0 = 1$, $X_1(j0) = 2T_1 = 4$ and $X_2(j0) = 2$, so $X_3(j0) = 2 \cdot 4 + 2 = 10$. In time, $x_1(t-4)$ is 1 on $2 < t < 6$ and $x_2(t-3)$ is 1 on $2 < t < 4$. So x_3 is $2 + 1 = 3$ on $2 < t < 4$ and 2 on $4 < t < 6$, with area $3 \cdot 2 + 2 \cdot 2 = 10$. The two values agree.

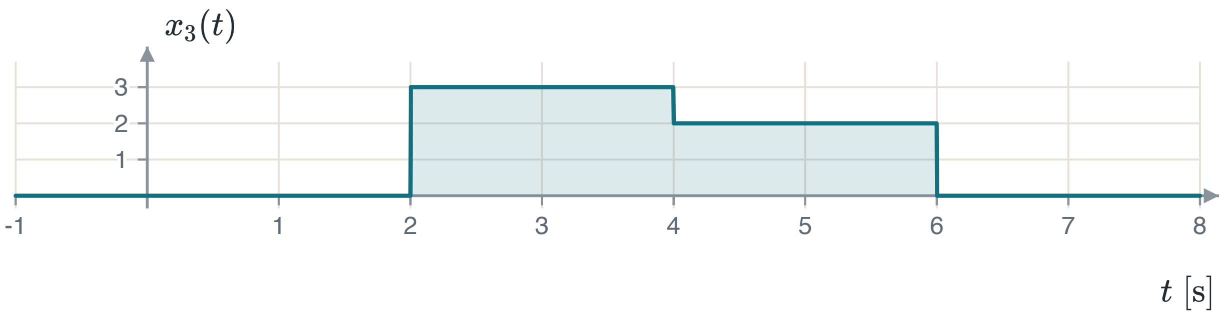

FIGURE 5.15 $x_3(t)$ is 3 on $2 < t < 4$ and 2 on $4 < t < 6$. The shaded area is 10.

Frequency shift

Start from the expression the property states, $e^{+j\omega_0 t}x(t)$, where ω_0 is a fixed frequency. Combine the two exponentials before anything else. The exponent becomes $-j(\omega - \omega_0)t$, which is the analysis integral read at $\omega - \omega_0$, and no substitution is needed:

$$\begin{aligned}\mathcal{F}\{e^{j\omega_0 t}x(t)\} &= \int_{-\infty}^{\infty} e^{j\omega_0 t}x(t)\,e^{-j\omega t}\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty} x(t)\,e^{-j(\omega-\omega_0)t}\,\mathrm{d}t \\ &= X\big(j(\omega-\omega_0)\big).\end{aligned}$$

The whole spectrum moves by ω_0. If $Y(j\omega) = 1$ on $|\omega| < 2\pi$, then $e^{j\pi t}y(t)$ has its band on $-\pi < \omega < 3\pi$. The operand is a complex exponential in **time** with a fixed frequency ω_0. The time-shift factor $e^{-j\omega t_0}$ has a fixed **time** and multiplies the spectrum; opening this proof with it proves a different statement.

Conjugation and the symmetry of a real signal

Conjugation. The conjugate of an integral is the integral of the conjugate, and conjugating a product conjugates each factor. The conjugate of $e^{-j\omega t}$ is $e^{+j\omega t}$, which is the analysis kernel evaluated at $-\omega$:

$$\begin{aligned}\mathcal{F}\{x^*(t)\} &= \int_{-\infty}^{\infty} x^*(t)\,e^{-j\omega t}\,\mathrm{d}t \\ &= \left[\int_{-\infty}^{\infty} x(t)\,e^{+j\omega t}\,\mathrm{d}t\right]^* \\ &= \left[\int_{-\infty}^{\infty} x(t)\,e^{-j(-\omega)t}\,\mathrm{d}t\right]^* = \big[X(-j\omega)\big]^* = X^*(-j\omega).\end{aligned}$$

Time reversal. Put $x(-t)$ into the analysis equation and substitute $\tau = -t$, so $t = -\tau$ and $\mathrm{d}t = -\mathrm{d}\tau$. As t runs from $-\infty$ to ∞, τ runs from ∞ to $-\infty$, so the limits arrive reversed. Swapping them back absorbs the minus sign of $\mathrm{d}t = -\mathrm{d}\tau$:

$$\begin{aligned}\mathcal{F}\{x(-t)\} &= \int_{-\infty}^{\infty} x(-t)\, e^{-j\omega t}\,\mathrm{d}t \\ &= \int_{\infty}^{-\infty} x(\tau)\, e^{-j\omega(-\tau)}\,(-\mathrm{d}\tau) \\ &= \int_{-\infty}^{\infty} x(\tau)\, e^{-j(-\omega)\tau}\,\mathrm{d}\tau = X(-j\omega).\end{aligned}$$

A real signal. If x is real then $x^*(t) = x(t)$, so the two sides of the conjugation property have the same transform: $X(j\omega) = X^*(-j\omega)$, or, conjugating both sides, $X(-j\omega) = X^*(j\omega)$. Everything follows from that one line. Write $X(j\omega) = R(\omega) + jI(\omega)$ with R and I real. Then $X(-j\omega) = R(-\omega) + jI(-\omega)$ and $X^*(j\omega) = R(\omega) - jI(\omega)$. Equating real parts and imaginary parts gives $R(-\omega) = R(\omega)$ and $I(-\omega) = -I(\omega)$: the real part of X is even and the imaginary part is odd. Taking modulus and angle of the same equation, $|X(-j\omega)| = |X^*(j\omega)| = |X(j\omega)|$ and $\angle X(-j\omega) = \angle X^*(j\omega) = -\angle X(j\omega)$: the magnitude is even and the phase is odd. For example, if x is real and $X(j2) = 3 - 4j$, then $X(-j2) = 3 + 4j$.

A real and even signal. Evenness gives $x(-t) = x(t)$, so by time reversal $X(-j\omega) = X(j\omega)$. Realness gives $X(-j\omega) = X^*(j\omega)$. Together, $X(j\omega) = X^*(j\omega)$, so X is real; and $X(-j\omega) = X(j\omega)$ says it is even. A real and odd signal has $x(-t) = -x(t)$, so $X(-j\omega) = -X(j\omega)$. With $X(-j\omega) = X^*(j\omega)$ this gives $X^*(j\omega) = -X(j\omega)$, so X is purely imaginary, and odd.

> ⚠ REAL DOES NOT MEAN ZERO PHASE
>
> A real transform can be negative. Where $X(j\omega) < 0$ the magnitude is $-X$ and the phase is π, not 0. Only a real and non-negative transform has zero phase everywhere. The sinc of the rectangular pulse is the standard counterexample: it is real, and its side lobes are negative in turn.

Even and odd parts

For a real signal, transform the even part $\mathcal{E}v\{x\} = \frac{1}{2}[x(t) + x(-t)]$ by linearity. Time reversal gives $x(-t) \leftrightarrow X(-j\omega)$, and for a real signal $X(-j\omega) = X^*(j\omega)$. Adding a complex number to its conjugate leaves twice the real part:

$$\begin{aligned}\mathcal{F}\{\mathcal{E}v\{x\}\} &= \tfrac{1}{2}\big[X(j\omega) + X(-j\omega)\big] = \tfrac{1}{2}\big[X(j\omega) + X^*(j\omega)\big] = \mathrm{Re}\{X(j\omega)\}, \\ \mathcal{F}\{\mathcal{O}dd\{x\}\} &= \tfrac{1}{2}\big[X(j\omega) - X(-j\omega)\big] = \tfrac{1}{2}\big[X(j\omega) - X^*(j\omega)\big] = j\,\mathrm{Im}\{X(j\omega)\}.\end{aligned}$$

The second line uses $X - X^* = 2j\,\mathrm{Im}\{X\}$. Splitting a real signal into even and odd parts splits its spectrum into real and imaginary parts. The two symmetry rules above are the cases in which one part is absent: a real, odd signal has no even part, so $X = j\,\mathrm{Im}\{X\}$ is imaginary and odd. Check it on $x(t) = e^{-at}u(t)$. Its even part is $\frac{1}{2}[e^{-at}u(t) + e^{at}u(-t)] = \frac{1}{2}e^{-a|t|}$, and its odd part is $\frac{1}{2}[e^{-at}u(t) - e^{at}u(-t)] = \frac{1}{2}\,\mathrm{sgn}(t)e^{-a|t|}$. Transform each part from the pairs already found. The second uses $e^{at}u(-t) \leftrightarrow 1/(a - j\omega)$, which is Example 5.3 reversed in time. Then compare with the real and imaginary parts of $1/(a + j\omega)$, obtained by multiplying numerator and denominator by $a - j\omega$:

$$\begin{aligned}
\tfrac{1}{2}e^{-a|t|} &\longleftrightarrow \tfrac{1}{2}\cdot\frac{2a}{a^2+\omega^2} = \frac{a}{a^2+\omega^2},\\
\tfrac{1}{2}\,\mathrm{sgn}(t)e^{-a|t|} &\longleftrightarrow \tfrac{1}{2}\left[\frac{1}{a+j\omega}-\frac{1}{a-j\omega}\right] = \tfrac{1}{2}\cdot\frac{-2j\omega}{a^2+\omega^2} = \frac{-j\omega}{a^2+\omega^2},
\end{aligned}$$

$$\frac{1}{a+j\omega} = \frac{a-j\omega}{(a+j\omega)(a-j\omega)} = \frac{a}{a^2+\omega^2} + j\,\frac{-\omega}{a^2+\omega^2}.$$

The even part gives exactly $\mathrm{Re}\{1/(a+j\omega)\}$ and the odd part gives exactly $j\,\mathrm{Im}\{1/(a+j\omega)\}$, as the property says.

Differentiation in frequency

Differentiate the analysis integral with respect to ω. Now t is the variable of integration and is held fixed, and the derivative of $e^{-j\omega t}$ with respect to ω is $-jt\,e^{-j\omega t}$:

$$\begin{aligned}\frac{\mathrm{d}X(j\omega)}{\mathrm{d}\omega} &= \int_{-\infty}^{\infty} x(t)\,\frac{\partial}{\partial\omega}e^{-j\omega t}\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty}\big[-jt\,x(t)\big]e^{-j\omega t}\,\mathrm{d}t = \mathcal{F}\{-jt\,x(t)\}.\end{aligned}$$

So $\mathrm{d}X/\mathrm{d}\omega$ is the transform of $-jt\,x(t)$. Multiply both sides by j and use $j\cdot(-j)=1$: $j\,\mathrm{d}X/\mathrm{d}\omega = \mathcal{F}\{t\,x(t)\}$, which is the property $t\,x(t) \leftrightarrow j\,\mathrm{d}X(j\omega)/\mathrm{d}\omega$. It gives the pair for $te^{-at}u(t)$ from the pair for $e^{-at}u(t)$ without another transform integral. Write $1/(a+j\omega) = (a+j\omega)^{-1}$ and use the chain rule, with $\mathrm{d}(a+j\omega)/\mathrm{d}\omega = j$:

$$\begin{aligned}\mathcal{F}\{te^{-at}u(t)\} &= j\,\frac{\mathrm{d}}{\mathrm{d}\omega}\left[\frac{1}{a+j\omega}\right] = j\cdot\frac{-1\cdot j}{(a+j\omega)^2} \\ &= \frac{-j^2}{(a+j\omega)^2} = \frac{1}{(a+j\omega)^2}, \qquad a>0.\end{aligned}$$

Check it at $\omega=0$ with $a=1$. The pair gives $1/(1+j0)^2 = 1$. The area of $te^{-t}u(t)$, by parts with the antiderivative $-(t+1)e^{-t}$, is $\big[-(t+1)e^{-t}\big]_0^{\infty} = 0-(-1) = 1$. The signal $te^{-t}u(t)$ starts at 0 and peaks at $t=1$, where it equals $e^{-1}\approx 0.37$.

Differentiation in time

Differentiate the **synthesis equation**, not the signal. On the right, the only factor that depends on t is $e^{j\omega t}$, and ω is the variable of integration, so it is held fixed. The derivative of $e^{j\omega t}$ with respect to t is $j\omega e^{j\omega t}$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}x(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)\,e^{j\omega t}\,\mathrm{d}\omega\right] \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)\,\frac{\partial}{\partial t}e^{j\omega t}\,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\big[j\omega X(j\omega)\big]e^{j\omega t}\,\mathrm{d}\omega.\end{aligned}$$

The last line is the synthesis equation applied to the function $j\omega X(j\omega)$. So $j\omega X(j\omega)$ is the transform of $\mathrm{d}x/\mathrm{d}t$. If $X(j2)=0.5$, for example, the transform of $\mathrm{d}x/\mathrm{d}t$ at $\omega=2$ is $j2\cdot 0.5 = j$. Differentiating again multiplies by another $j\omega$, and after n derivatives:

$$\frac{\mathrm{d}^n x}{\mathrm{d}t^n} \longleftrightarrow (j\omega)^n X(j\omega).$$

The factor $j\omega$ removes low frequencies and lifts high ones. It also turns a differential equation into an algebraic equation in $j\omega$, which Section 5.6 uses.

DO NOT MIX THE TWO DOMAINS

The equation $\mathrm{d}x/\mathrm{d}t = j\omega\, x(t)$ is false. The frequency ω is the variable of integration in the synthesis equation; it is not a constant of the time-domain signal. For $x(t) = e^{-t^2}$, $\mathrm{d}x/\mathrm{d}t = -2te^{-t^2}$ is $-2e^{-1} = -0.735759$ at $t = 1$, while $j\omega\, x(t)$ at $t = 1$, $\omega = 3$ is $3e^{-1}j = 1.103638j$. The differentiation property is correct only in the form derived from the synthesis integral.

EXAMPLE 5.13 — THE UNIT STEP

GIVEN $u(t)$, which is 1 for $t > 0$ and 0 for $t < 0$. It fails both existence conditions.

FIND $U(j\omega)$.

METHOD Split the step into its mean and an odd part: $u(t) = \frac{1}{2} + \frac{1}{2}\,\mathrm{sgn}(t)$, where $\mathrm{sgn}(t)$ is $+1$ for $t > 0$ and -1 for $t < 0$. The mean has a known transform. Obtain the transform of $\mathrm{sgn}(t)$ as the limit of $e^{-a|t|}\,\mathrm{sgn}(t)$ as $a \to 0$, a signal that has an ordinary transform.

SOLUTION For $a > 0$ the signal $e^{-a|t|}\,\mathrm{sgn}(t)$ equals $e^{-at}u(t)$ for $t > 0$ and $-e^{at}u(-t)$ for $t < 0$. Example 5.3 gives $e^{-at}u(t) \leftrightarrow 1/(a + j\omega)$. The signal $e^{at}u(-t)$ is that signal reversed in time, so by time reversal its transform is $1/(a - j\omega)$. Linearity, then a common denominator:

$$\begin{aligned}\mathcal{F}\{e^{-a|t|}\,\mathrm{sgn}(t)\} &= \frac{1}{a + j\omega} - \frac{1}{a - j\omega} \\ &= \frac{(a - j\omega) - (a + j\omega)}{(a + j\omega)(a - j\omega)} \\ &= \frac{-2j\omega}{a^2 + \omega^2}.\end{aligned}$$

Let $a \to 0$ at a fixed $\omega \neq 0$, and use $-j = 1/j$:

$$\lim_{a\to 0} \frac{-2j\omega}{a^2 + \omega^2} = \frac{-2j\omega}{\omega^2} = \frac{-2j}{\omega} = \frac{2}{j\omega}.$$

At $\omega = 0$ the transform is 0 for every a, and it is odd in ω, so no impulse forms at the origin in the limit. Hence $\mathrm{sgn}(t) \leftrightarrow 2/(j\omega)$. The mean gives $\frac{1}{2} \leftrightarrow \frac{1}{2} \cdot 2\pi\delta(\omega) = \pi\delta(\omega)$. Add the two parts:

$$u(t) \quad \longleftrightarrow \quad \pi\delta(\omega) + \tfrac{1}{2} \cdot \frac{2}{j\omega} = \frac{1}{j\omega} + \pi\delta(\omega).$$

CHECK $\mathrm{d}u/\mathrm{d}t = \delta(t)$, whose transform is 1. The differentiation property multiplies U by $j\omega$:

$$j\omega\left[\frac{1}{j\omega} + \pi\delta(\omega)\right] = 1 + j\pi\,\omega\delta(\omega) = 1,$$

because $\omega\delta(\omega) = 0 \cdot \delta(\omega) = 0$. The same split gives the reversed step: $u(-t) = \frac{1}{2} - \frac{1}{2}\,\mathrm{sgn}(t) \leftrightarrow \pi\delta(\omega) - 1/(j\omega)$. The mean is still $\frac{1}{2}$; only the odd part changes sign.

Integration, and the impulse it leaves behind

The running integral is a convolution with the unit step. In the convolution integral, the factor $u(t - \tau)$ equals 1 for $\tau < t$ and 0 for $\tau > t$, so it cuts the upper limit at t:

$$x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau)\, u(t - \tau)\, \mathrm{d}\tau = \int_{-\infty}^{t} x(\tau)\, \mathrm{d}\tau.$$

The convolution property of Section 5.5 multiplies the transforms, and Example 5.13 gives the transform of the step. In the product, $X(j\omega)\delta(\omega) = X(j0)\delta(\omega)$, because the impulse is zero away from $\omega = 0$ and only the value of X at the origin survives. Writing $X(0)$ for $X(j0)$:

$$\begin{aligned}\mathcal{F}\Big\{\int_{-\infty}^{t} x(\tau)\,\mathrm{d}\tau\Big\} &= X(j\omega)\Big[\frac{1}{j\omega} + \pi\delta(\omega)\Big] \\ &= \frac{1}{j\omega}X(j\omega) + \pi X(0)\,\delta(\omega).\end{aligned}$$

INCLUDE THE IMPULSE TERM WHEN THE AREA IS NON-ZERO

$X(0) = \int x(t)\,\mathrm{d}t$ is the total area of the signal. If this area is non-zero, the running integral settles at a non-zero constant, and a constant has an impulse at $\omega = 0$. Writing only $X(j\omega)/(j\omega)$ omits that constant. For $x(t) = 1$ on $0 < t < 2$ the area is $X(0) = 2$, so the impulse term is $2\pi\,\delta(\omega)$. A pulse of area zero, such as $+1$ on $0 < t < 1$ followed by -1 on $1 < t < 2$, has a running integral that returns to 0, and no impulse term.

EXAMPLE 5.14 – A TRANSFORM BY DIFFERENTIATION

GIVEN $x(t) = 1$ for $|t| \le 1$, $x(t) = 2 - |t|$ for $1 < |t| < 2$, and 0 elsewhere: a trapezoid.

FIND $X(j\omega)$.

METHOD The ramps make the analysis integral long. Differentiate instead: $g = \mathrm{d}x/\mathrm{d}t$ is two pulses, whose transforms are known. Since $x = 0$ for $t < -2$, x is the running integral of g, and the integration property returns X from G.

SOLUTION Differentiate piece by piece. On $-2 < t < -1$, $x = 2 + t$, so $g = 1$. On $|t| < 1$, $x = 1$, so $g = 0$. On $1 < t < 2$, $x = 2 - t$, so $g = -1$. Elsewhere $g = 0$. Let $p(t)$ be the pulse of unit width, 1 on $|t| < \frac{1}{2}$. Then $g(t) = p(t + 1.5) - p(t - 1.5)$. Example 5.5 with $T_1 = \frac{1}{2}$ gives $p(t) \leftrightarrow 2\sin(\omega/2)/\omega$. The time shift gives $p(t + 1.5) \leftrightarrow e^{j1.5\omega}P(j\omega)$ and $p(t - 1.5) \leftrightarrow e^{-j1.5\omega}P(j\omega)$. Linearity, then Euler's $e^{j\theta} - e^{-j\theta} = 2j\sin\theta$ with $\theta = 1.5\omega$:

$$\begin{aligned}G(j\omega) &= \big(e^{j1.5\omega} - e^{-j1.5\omega}\big)\frac{2\sin(\omega/2)}{\omega} \\ &= 2j\sin(1.5\omega)\cdot\frac{2\sin(\omega/2)}{\omega} = \frac{4j\sin(1.5\omega)\sin(\omega/2)}{\omega}.\end{aligned}$$

The area of g is $G(0) = 1 - 1 = 0$: one pulse up, one down. So the integration property leaves no impulse term:

$$\begin{aligned}X(j\omega) &= \frac{G(j\omega)}{j\omega} + \pi G(0)\,\delta(\omega) \\ &= \frac{4j\sin(1.5\omega)\sin(\omega/2)}{j\omega\cdot\omega} + 0 = \frac{4\sin(1.5\omega)\sin(\omega/2)}{\omega^2}.\end{aligned}$$

CHECK For small ω, $\sin(1.5\omega) \approx 1.5\omega$ and $\sin(\omega/2) \approx \omega/2$, so $X(j0) = 4\cdot 1.5\cdot 0.5 = 3$. The trapezoid has parallel sides 2 and 4 and height 1, so its area is $\frac{1}{2}(2 + 4)\cdot 1 = 3$. X is real and even, as x is. A second route: the trapezoid is the convolution of the pulse of half-width 1.5 with the pulse of half-width 0.5, because their overlap has length 1 for $|t| \le 1$ and shrinks linearly to 0 at $|t| = 2$. The convolution property (Section 5.5) multiplies their transforms: $\frac{2\sin(1.5\omega)}{\omega} \cdot \frac{2\sin(0.5\omega)}{\omega}$, the same result. At $\omega = \pi$ it gives $4\sin(1.5\pi)\sin(\pi/2)/\pi^2 = -4/\pi^2 = -0.405285$.

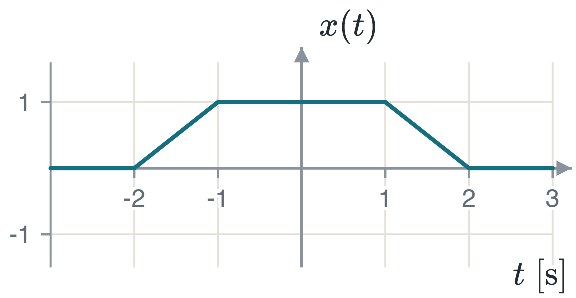

FIGURE 5.16 The trapezoid $x(t)$.

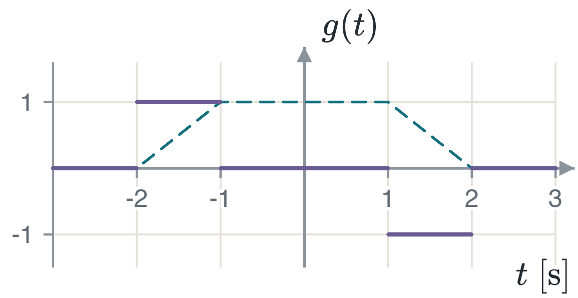

FIGURE 5.17 $g = \mathrm{d}x/\mathrm{d}t$: pulses of height ± 1 where x ramps.

Time scaling

Put $x(at)$ into the analysis equation and substitute $\tau = at$, so $t = \tau/a$ and $\mathrm{d}t = \mathrm{d}\tau/a$. For $a > 0$ the limits keep their order, since $\tau \to \pm\infty$ when $t \to \pm\infty$, and the coefficient is $1/a = 1/|a|$:

$$\begin{aligned}\mathcal{F}\{x(at)\} &= \int_{-\infty}^{\infty} x(at)\, e^{-j\omega t}\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau/a}\,\frac{\mathrm{d}\tau}{a} \\ &= \frac{1}{a}\int_{-\infty}^{\infty} x(\tau)\, e^{-j(\omega/a)\tau}\,\mathrm{d}\tau = \frac{1}{a}X\Big(j\frac{\omega}{a}\Big), \qquad a > 0.\end{aligned}$$

For $a < 0$ the substitution sends $t \to -\infty$ to $\tau \to +\infty$ and $t \to +\infty$ to $\tau \to -\infty$, so the limits arrive reversed. Swapping them back costs one minus sign, and $-1/a = 1/|a|$ for negative a. That is the only sign in the calculation:

$$\begin{aligned}\mathcal{F}\{x(at)\} &= \int_{+\infty}^{-\infty} x(\tau)\, e^{-j(\omega/a)\tau}\,\frac{\mathrm{d}\tau}{a} \\ &= -\frac{1}{a}\int_{-\infty}^{\infty} x(\tau)\, e^{-j(\omega/a)\tau}\,\mathrm{d}\tau = \frac{1}{|a|}X\Big(j\frac{\omega}{a}\Big), \qquad a < 0.\end{aligned}$$

Both cases read $x(at) \leftrightarrow \frac{1}{|a|}X(j\omega/a)$ for $a \neq 0$. The modulus sits on a in the factor, not in the argument. Compress a signal in time ($a > 1$) and its spectrum widens by a and drops by $1/a$; stretch it and the spectrum narrows. This is the inverse relation of Section 5.2. At $\omega = 0$ the property reads $\frac{1}{|a|}X(j0)$: if $X(j0) = 6$, the transform of $x(3t)$ at $\omega = 0$ is 2, because the area of $x(3t)$ is a third of the area of x.

COUNTING THE FLIP TWICE

Writing the reversed limits **and** an explicit -1 in front applies the same correction twice and gives $-\frac{1}{|a|}X(j\omega/a)$, which contradicts the property. Do the flip once: reverse the limits, or write the minus, never both. The case $a = -1$ is time reversal, $x(-t) \leftrightarrow X(-j\omega)$, with coefficient 1. For $x(t) = e^{-t}u(t) \leftrightarrow 1/(1 + j\omega)$ it gives $x(-t) \leftrightarrow 1/(1 - j\omega)$.

EXAMPLE 5.15 – SCALING A BAND

GIVEN $X(j\omega) = 1$ on $|\omega| < 2\pi$ and 0 elsewhere.

FIND The transforms of $x(0.5t)$ and $x(2t)$, with heights and band edges.

METHOD Apply $x(at) \leftrightarrow \frac{1}{|a|}X(j\omega/a)$ once for $a = 0.5$ and once for $a = 2$. Then solve each band condition for ω.

SOLUTION For $a = 0.5$: $x(0.5t) \leftrightarrow \frac{1}{0.5}X(j\omega/0.5) = 2X(j2\omega)$. It is non-zero where $|2\omega| < 2\pi$, that is $|\omega| < \pi$, and there it equals $2 \cdot 1 = 2$. For $a = 2$: $x(2t) \leftrightarrow \frac{1}{2}X(j\omega/2)$. It is non-zero where $|\omega/2| < 2\pi$, that is $|\omega| < 4\pi$, and there it equals 0.5. So

$$\begin{aligned} x(0.5t) &\longleftrightarrow 2 \text{ on } |\omega| < \pi, \\ x(2t) &\longleftrightarrow 0.5 \text{ on } |\omega| < 4\pi. \end{aligned}$$

CHECK The areas of the three spectra are $2 \cdot 2\pi$, $1 \cdot 4\pi$ and $0.5 \cdot 8\pi$, all equal to 4π. At $t = 0$ the synthesis equation reads $x(0) = \frac{1}{2\pi}\int X \, \mathrm{d}\omega$, so all three signals have the value $4\pi/2\pi = 2$ at the origin. That is what a time scaling cannot change: $x(a \cdot 0) = x(0)$.

Duality

The two equations of the pair differ only by a sign and a factor, so any pair can be read a second time with the domains exchanged.

DUALITY

$$x(t) \longleftrightarrow X(j\omega) \qquad \Longrightarrow \qquad X(t) \longleftrightarrow 2\pi\, x(-\omega)$$

Here $X(t)$ means the formula for $X(j\omega)$ with ω replaced by t. The argument on the right is $-\omega$, a real number, and not $-j\omega$: the letter x names a signal, and a signal takes a real argument. The j belongs to the frequency-domain function only, and duality is where that distinction is most easily lost.

Proof. Start from the synthesis equation and multiply by 2π:

$$2\pi\, x(t) = \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, \mathrm{d}\omega.$$

The names of the variables carry no meaning, so rename them: call the variable of integration t and the free variable ω. The equation becomes

$$2\pi\, x(\omega) = \int_{-\infty}^{\infty} X(jt)\, e^{j\omega t}\, \mathrm{d}t.$$

This holds for every real ω, so it holds with ω replaced by $-\omega$. The exponent $e^{j(-\omega)t} = e^{-j\omega t}$ is now the analysis kernel:

$$2\pi\, x(-\omega) = \int_{-\infty}^{\infty} X(jt)\, e^{-j\omega t}\, \mathrm{d}t = \mathcal{F}\{X(t)\}.$$

The right side is the analysis equation applied to the time signal $X(t)$, so its transform is $2\pi x(-\omega)$. Every pair therefore gives a second pair at no cost. From $e^{-|t|} \leftrightarrow 2/(1+\omega^2)$ (Example 5.4 with $a = 1$), duality gives $2/(1+t^2) \leftrightarrow 2\pi e^{-|-\omega|} = 2\pi e^{-|\omega|}$. Duality also carries properties across: the time-shift rule and the frequency-shift rule are one rule read twice, and so are differentiation in time and differentiation in frequency.

EXAMPLE 5.16 — DUALITY ON THE RECTANGULAR PULSE

GIVEN $x_1(t) = 1$ on $|t| < W$ and 0 elsewhere, so $X_1(j\omega) = 2\sin(W\omega)/\omega$ by Example 5.5.

FIND The transform of $x_2(t) = 2\sin(Wt)/t$.

METHOD x_2 has the same formula as X_1 with the independent variable renamed, so duality applies. Use duality first, then confirm the result with the synthesis equation.

SOLUTION The signal $x_2(t) = 2\sin(Wt)/t$ is $X_1(j\omega)$ with ω replaced by t, that is $x_2(t) = X_1(t)$ in the notation of the duality box. Duality then gives $X_2(j\omega) = 2\pi x_1(-\omega)$. The pulse x_1 is even, so $x_1(-\omega) = x_1(\omega)$, which is 1 on $|\omega| < W$ and 0 beyond:

$$X_2(j\omega) = 2\pi\, x_1(-\omega) = 2\pi\, x_1(\omega) = \begin{cases} 2\pi, & |\omega| < W \\ 0, & |\omega| > W. \end{cases}$$

CHECK The second route works backwards. Put a band of height 2π on $|\omega| < W$ through the synthesis equation; for $t \neq 0$ the antiderivative of $e^{j\omega t}$ in ω is $e^{j\omega t}/(jt)$:

$$\begin{aligned}\frac{1}{2\pi}\int_{-W}^{W} 2\pi\, e^{j\omega t}\, \mathrm{d}\omega = \left[\frac{e^{j\omega t}}{jt}\right]_{-W}^{W} &= \frac{e^{jWt} - e^{-jWt}}{jt} \\ &= \frac{2j\sin(Wt)}{jt} = \frac{2\sin(Wt)}{t},\end{aligned}$$

which is x_2. The two routes use different equations, so their agreement is a real check. At the origin, $X_2(j0) = 2\pi = 6.283185$ for any W. In time, $x_2(0) = \lim_{t\to 0} 2W\cos(Wt)/1 = 2W$ by l'Hôpital; with the unnormalised sinc, $\operatorname{sinc}(\theta) = \sin\theta/\theta$, $x_2(t) = 2W\operatorname{sinc}(Wt)$.

Parseval's relation

Energy and power in this course are normalised: every signal is a voltage across $R = 1\,\Omega$, so the instantaneous power is $|x(t)|^2$ and the energy is its integral, in joules.

PARSEVAL'S RELATION

$$E_\infty = \int_{-\infty}^{\infty} |x(t)|^2\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2\,\mathrm{d}\omega$$

The quantity $|X(j\omega)|^2$ is the **energy spectral density**: $\frac{1}{2\pi}|X(j\omega)|^2\,\mathrm{d}\omega$ is the energy in a narrow band. The $1/2\pi$ sits on the frequency side, as in the synthesis equation.

Proof. Write $|x(t)|^2 = x(t)\,x^*(t)$. Replace $x^*(t)$ by the conjugate of the synthesis equation; conjugating turns $X(j\omega)$ into $X^*(j\omega)$ and $e^{j\omega t}$ into $e^{-j\omega t}$. Then exchange the order of the two integrals, so that the t integral is done first. The inner integral is the analysis equation and equals $X(j\omega)$:

$$\begin{aligned}
\int_{-\infty}^{\infty} |x(t)|^2\,\mathrm{d}t &= \int_{-\infty}^{\infty} x(t)\,x^*(t)\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} x(t)\Big[\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)\,e^{-j\omega t}\,\mathrm{d}\omega\Big]\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)\Big[\int_{-\infty}^{\infty} x(t)\,e^{-j\omega t}\,\mathrm{d}t\Big]\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)\,X(j\omega)\,\mathrm{d}\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2\,\mathrm{d}\omega.
\end{aligned}$$

For a real signal, $|x(t)|^2 = x^2(t)$, and that is the only simplification available. The claim $|x(t)| = x(t)$ is stronger and fails wherever a real signal is negative.

The relation can be tried on $e^{-at}u(t)$, $a > 0$, in both domains. In time, the step cuts the lower limit at 0 and $|e^{-at}|^2 = e^{-2at}$:

$$E_\infty = \int_0^{\infty} e^{-2at}\,\mathrm{d}t = \left[\frac{e^{-2at}}{-2a}\right]_0^{\infty} = \frac{0-1}{-2a} = \frac{1}{2a}.$$

In frequency, $|X(j\omega)|^2 = 1/(a^2+\omega^2)$ from Example 5.3. Substitute $\omega = au$, so $\mathrm{d}\omega = a\,\mathrm{d}u$ and the limits stay infinite; then use the antiderivative $\tan^{-1} u$:

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{a^2+\omega^2} &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{a\,\mathrm{d}u}{a^2(1+u^2)} = \frac{1}{2\pi a}\big[\tan^{-1} u\big]_{-\infty}^{\infty} \\
&= \frac{1}{2\pi a}\Big[\frac{\pi}{2} - \Big(-\frac{\pi}{2}\Big)\Big] = \frac{1}{2\pi}\cdot\frac{\pi}{a} = \frac{1}{2a}.
\end{aligned}$$

The two routes agree: 0.5 J for $a = 1$, and 0.25 J for $a = 2$. Without the $1/2\pi$ the second route would report 2π times too much.

EXAMPLE 5.17 — ENERGY FROM A TWO-BAND SPECTRUM

GIVEN $X_3(j\omega) = 2$ for $|\omega| < 2\pi$, 1 for $2\pi < |\omega| < 4\pi$, and 0 beyond, with $R = 1\,\Omega$.

FIND The total energy of x_3.

METHOD The spectrum is piecewise constant, so the frequency-domain energy integral is simpler than the time-domain one. Apply Parseval, square each height, and multiply by the width of its piece.

SOLUTION Split the frequency integral of Parseval's relation at the points where the height changes. On each piece $|X_3|^2$ is a constant, so the integral is that constant times the width of the piece:

$$\begin{aligned}
E_\infty &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X_3(j\omega)|^2\, \mathrm{d}\omega \\
&= \frac{1}{2\pi}\Big[\int_{-4\pi}^{-2\pi} 1^2\, \mathrm{d}\omega + \int_{-2\pi}^{2\pi} 2^2\, \mathrm{d}\omega + \int_{2\pi}^{4\pi} 1^2\, \mathrm{d}\omega\Big] \\
&= \frac{1}{2\pi}\big[1\cdot 2\pi + 4\cdot 4\pi + 1\cdot 2\pi\big] \\
&= \frac{1}{2\pi}\big[2\pi + 16\pi + 2\pi\big] = \frac{20\pi}{2\pi} = 10\ \mathrm{J}.
\end{aligned}$$

The inner band contributes $2^2 \cdot 4\pi = 16\pi$ and the two outer bands $1^2 \cdot 2\pi$ each.

CHECK

Write X_3 as the sum of two ideal low-pass bands of height 1, one on $|\omega| < 4\pi$ and one on $|\omega| < 2\pi$; where they overlap the heights add to 2. Example 5.6 with $W = 4\pi$ and $W = 2\pi$ then gives, by linearity,

$$x_3(t) = \frac{\sin(4\pi t)}{\pi t} + \frac{\sin(2\pi t)}{\pi t} = \frac{\sin(2\pi t) + \sin(4\pi t)}{\pi t}.$$

Its peak is $x_3(0) = 4 + 2 = 6$. Using the heights unsquared gives $\frac{1}{2\pi}\int X_3\, \mathrm{d}\omega = \frac{1}{2\pi}[2\pi + 8\pi + 2\pi] = 6$: that is the peak $x_3(0)$, not the energy. Square the height before integrating.

5.5 Convolution and multiplication

THE CONVOLUTION PROPERTY

$$y(t) = x(t) * h(t) \;\longleftrightarrow\; Y(j\omega) = X(j\omega)\, H(j\omega)$$

The statement uses one set of symbols: $x \leftrightarrow X$ is the input, $h \leftrightarrow H$ the impulse response, and y the output. $H(j\omega)$ is the frequency response of the system. Using y both for a free second signal and for the output would make the statement refer to itself.

Proof. Write the convolution integral inside the analysis integral and exchange the order, so that the t integral is done first. The inner bracket is the transform of $h(t-\tau)$ with τ fixed, which the time-shift property gives as $e^{-j\omega\tau} H(j\omega)$. Then $H(j\omega)$ does not depend on τ and leaves the outer integral:

$$\begin{aligned}
Y(j\omega) &= \int_{-\infty}^{\infty}\Big[\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, \mathrm{d}\tau\Big] e^{-j\omega t}\, \mathrm{d}t \\
&= \int_{-\infty}^{\infty} x(\tau)\Big[\int_{-\infty}^{\infty} h(t-\tau)\, e^{-j\omega t}\, \mathrm{d}t\Big] \mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau} H(j\omega)\, \mathrm{d}\tau \\
&= H(j\omega)\int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau}\, \mathrm{d}\tau = H(j\omega)\, X(j\omega).
\end{aligned}$$

Convolution in time carries no factor. At $\omega = 0$ the property reads $Y(j0) = X(j0)H(j0)$: with $X(j0) = 4$ and $H(j0) = 0.5$, $Y(j0) = 2$. $H(j\omega)$ exists as an ordinary function when h is absolutely integrable, which for an LTI system is exactly bounded-input bounded-output stability.

Delay, differentiator, integrator

Three simple systems are each one operation of Section 5.4. The property of that operation gives $Y = HX$ directly, and H can be read off:

$$\begin{aligned} y(t) &= x(t-t_0) &&\implies && Y = e^{-j\omega t_0}X, && H(j\omega) = e^{-j\omega t_0}, \\ y(t) &= \frac{\mathrm{d}x}{\mathrm{d}t} &&\implies && Y = j\omega X, && H(j\omega) = j\omega, \\ y(t) &= \int_{-\infty}^{t} x(\tau)\,\mathrm{d}\tau &&\implies && Y = \Big[\frac{1}{j\omega} + \pi\delta(\omega)\Big]X, && H(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega). \end{aligned}$$

The third line uses $\pi X(0)\delta(\omega) = \pi\delta(\omega)X(j\omega)$. The integrator has impulse response $u(t)$, and its frequency response is the transform of the step (Example 5.13). The step is not absolutely integrable, so the integrator is not stable, and its H exists only in the limiting sense, with an impulse.

Read the magnitudes. A delay has $|H| = 1$: it changes only the phase, by $-\omega t_0$. A differentiator has $|H| = |\omega|$: it lifts high frequencies, so it also amplifies high-frequency noise. An integrator has $|H| = 1/|\omega|$ away from the origin: it does the opposite. For example, feed $x(t) = \cos(2t)$ to the differentiator. Its transform is $\pi\delta(\omega-2) + \pi\delta(\omega+2)$. Multiply by $j\omega$; each impulse keeps only the value of $j\omega$ at its own position:

$$\begin{aligned} Y(j\omega) &= j\omega\big[\pi\delta(\omega-2) + \pi\delta(\omega+2)\big] \\ &= j2\pi\,\delta(\omega-2) - j2\pi\,\delta(\omega+2) \\ &= -2\Big[\frac{\pi}{j}\,\delta(\omega-2) - \frac{\pi}{j}\,\delta(\omega+2)\Big]. \end{aligned}$$

The last line uses $\pi/j = -j\pi$. The bracket is the transform of $\sin(2t)$ (Example 5.9 with unit amplitude), so $y(t) = -2\sin(2t)$, of amplitude $|H(j2)| = 2$. Differentiating $\cos(2t)$ directly gives the same.

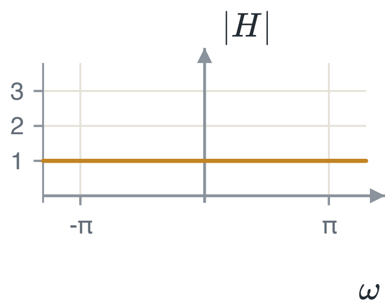

FIGURE 5.18 Delay: $|H| = 1$.

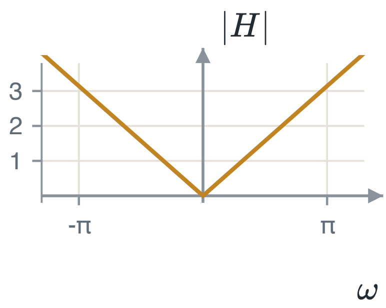

FIGURE 5.19 Differentiator: $|H| = |\omega|$.

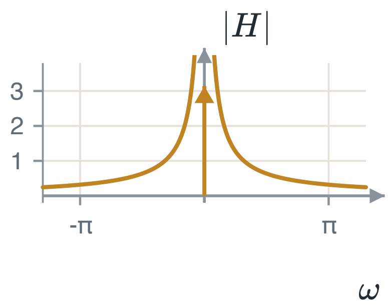

FIGURE 5.20 Integrator: $1/|\omega|$, and $\pi\delta(\omega)$.

EXAMPLE 5.18 — TWO EXPONENTIALS IN CASCADE

GIVEN $x(t) = e^{-at}u(t)$ and $h(t) = e^{-bt}u(t)$, with $a, b > 0$ and $a \neq b$.

FIND $y(t) = x * h$, and the magnitude $|Y(j\omega)|$ for $a = 1, b = 2$.

METHOD The output is a convolution in time, so it becomes multiplication in frequency. Multiply the transforms, expand the product into simple fractions, and invert each term.

SOLUTION Example 5.3 gives $X(j\omega) = 1/(a + j\omega)$ and $H(j\omega) = 1/(b + j\omega)$. Multiply them, then write the product as a sum of two simple fractions with unknown constants A and B:

$$Y(j\omega) = \frac{1}{(a + j\omega)(b + j\omega)} = \frac{A}{a + j\omega} + \frac{B}{b + j\omega}.$$

Multiply through by $(a + j\omega)(b + j\omega)$ to clear the denominators:

$$1 = A(b + j\omega) + B(a + j\omega).$$

This holds for every ω. Choose $j\omega = -a$ to remove the B term: $1 = A(b - a)$, so $A = 1/(b - a)$. Choose $j\omega = -b$ to remove the A term: $1 = B(a - b)$, so $B = 1/(a - b) = -A$. Hence

$$Y(j\omega) = \frac{1}{b - a}\left[\frac{1}{a + j\omega} - \frac{1}{b + j\omega}\right].$$

Each fraction is the pair of Example 5.3, so by linearity

$$y(t) = \frac{1}{b - a}\left[e^{-at}u(t) - e^{-bt}u(t)\right] = \frac{e^{-at} - e^{-bt}}{b - a}u(t).$$

For the magnitude, the modulus of a product is the product of the moduli. With $a = 1$ and $b = 2$:

$$|Y(j\omega)| = |X(j\omega)|\,|H(j\omega)| = \frac{1}{\sqrt{1 + \omega^2}} \cdot \frac{1}{\sqrt{4 + \omega^2}} = \frac{1}{\sqrt{(1 + \omega^2)(4 + \omega^2)}}.$$

The phases add in the same way: $\angle Y = \angle X + \angle H$.

CHECK $y(0) = (1 - 1)/(b - a) = 0$, as a convolution of two causal signals must be: they do not overlap at $t = 0$. For $a = 1, b = 2$, set $\mathrm{d}y/\mathrm{d}t = -e^{-t} + 2e^{-2t} = 0$, so $e^t = 2$ and the peak is at $t = \ln 2 = 0.693147$, with value $e^{-\ln 2} - e^{-2\ln 2} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$ exactly. At $\omega = 0$: $|X| = 1$, $|H| = 0.5$ and $|Y| = 0.5$; at $\omega = 1$: $|X| = 1/\sqrt{2}$, $|H| = 1/\sqrt{5}$ and $|Y| = 1/\sqrt{10} = 0.316228$. At each frequency the output magnitude is the product of the other two.

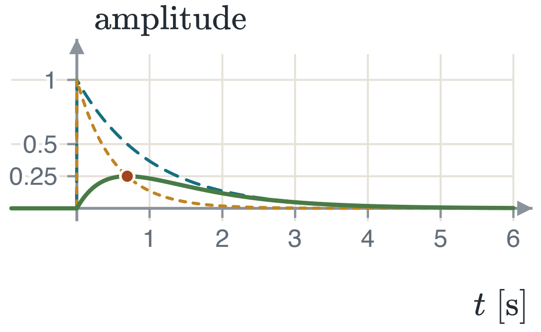

FIGURE 5.21 x (dashed), h (dotted) and y (solid) for $a = 1$, $b = 2$. The peak is 0.25 at $t = \ln 2$.

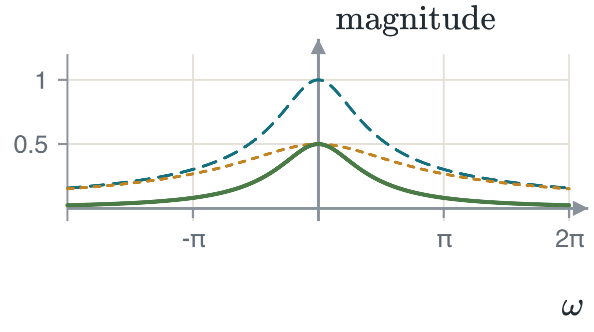

FIGURE 5.22 $|X|$ (dashed), $|H|$ (dotted) and their product $|Y|$ (solid).

EXAMPLE 5.19 – IDEAL FILTERS IN CASCADE

GIVEN $X(j\omega) = 2$ on $|\omega| \le 4\pi$ and an ideal low-pass system with $H(j\omega) = 3$ on $|\omega| \le 2\pi$, both zero beyond.

FIND $Y(j\omega)$, $y(t)$ and the three time-domain peaks.

METHOD An LTI system gives $Y = XH$. Multiply the input spectrum and the frequency response at each frequency, then invert with the pair of Example 5.6.

SOLUTION Multiply the two spectra frequency by frequency. On $|\omega| \le 2\pi$ both are non-zero: $Y = 2 \cdot 3 = 6$. On $2\pi < |\omega| \le 4\pi$ the input is 2 but the system is 0: $Y = 2 \cdot 0 = 0$. Beyond 4π both are zero. So $Y = 6$ on $|\omega| \le 2\pi$ and zero elsewhere; the narrower band decides. Invert with the synthesis equation, which is Example 5.6 with $W = 2\pi$ and an extra factor 6:

$$\begin{aligned} y(t) &= \frac{1}{2\pi}\int_{-2\pi}^{2\pi} 6\, e^{j\omega t}\, \mathrm{d}\omega = 6 \cdot \frac{1}{2\pi}\left[\frac{e^{j\omega t}}{jt}\right]_{-2\pi}^{2\pi} \\ &= 6 \cdot \frac{e^{j2\pi t} - e^{-j2\pi t}}{2\pi j t} = 6 \cdot \frac{2j\sin(2\pi t)}{2\pi j t} = \frac{6\sin(2\pi t)}{\pi t}. \end{aligned}$$

CHECK At $t = 0$ the synthesis integral is the area of the spectrum divided by 2π, since $e^{j\omega\cdot 0} = 1$. Each peak is therefore the area of its own band over 2π: $x(0) = 2 \cdot 8\pi/2\pi = 8$, $h(0) = 3 \cdot 4\pi/2\pi = 6$, $y(0) = 6 \cdot 4\pi/2\pi = 12$. The output peak is the largest because a peak counts area, not height, and $6 \times 4\pi$ exceeds $2 \times 8\pi$.

The multiplication property

MULTIPLICATION IN TIME

$$z(t) = x(t)\,y(t) \;\longleftrightarrow\; Z(j\omega) = \frac{1}{2\pi}\,X(j\omega) * Y(j\omega), \qquad X * Y = \int_{-\infty}^{\infty} X(j\theta)\,Y\big(j(\omega-\theta)\big)\,\mathrm{d}\theta$$

The convolution is over frequency. Convolution in time carries no factor; convolution in frequency carries $1/2\pi$. Without it the shape of Z is right and its height is 2π times too large.

Proof. Replace $y(t)$ in the analysis integral of $z = xy$ by its synthesis equation, written with the dummy variable θ so that it is not confused with the output frequency ω. Exchange the order of integration. The inner integral is the frequency-shift property, the analysis equation of x read at $\omega - \theta$:

$$\begin{aligned} Z(j\omega) &= \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} Y(j\theta)\,e^{j\theta t}\,\mathrm{d}\theta\right]e^{-j\omega t}\,\mathrm{d}t \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(j\theta)\left[\int_{-\infty}^{\infty} x(t)\,e^{-j(\omega-\theta)t}\,\mathrm{d}t\right]\mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(j\theta)\,X\big(j(\omega-\theta)\big)\,\mathrm{d}\theta = \frac{1}{2\pi}X(j\omega) * Y(j\omega). \end{aligned}$$

The last integral is the convolution of X and Y in the variable ω, and the $1/2\pi$ came in with the synthesis equation of y. Switching a signal on and off, modulating it by a carrier and cutting it to a window are all products in time. Under convolution the widths add: bands of half-widths B_1 and B_2 give a product whose spectrum has half-width $B_1 + B_2$. Half-widths π and 3π, for example, give 4π.

Amplitude modulation

A cosine of frequency ω_c is called a **carrier**. Its transform follows from Euler and the complex-exponential pair: $\cos(\omega_c t) = \frac{1}{2}e^{j\omega_c t} + \frac{1}{2}e^{-j\omega_c t} \leftrightarrow \pi\delta(\omega-\omega_c) + \pi\delta(\omega+\omega_c)$. Multiplying a signal by the carrier therefore convolves its spectrum with two impulses. Convolution with a shifted impulse shifts the function, $X(j\omega) * \delta(\omega-\omega_c) = X\big(j(\omega-\omega_c)\big)$, so:

$$\begin{aligned} Z(j\omega) &= \frac{1}{2\pi}X(j\omega) * \big[\pi\delta(\omega-\omega_c) + \pi\delta(\omega+\omega_c)\big] \\ &= \frac{\pi}{2\pi}X\big(j(\omega-\omega_c)\big) + \frac{\pi}{2\pi}X\big(j(\omega+\omega_c)\big) \\ &= \tfrac{1}{2}X\big(j(\omega-\omega_c)\big) + \tfrac{1}{2}X\big(j(\omega+\omega_c)\big). \end{aligned}$$

The factor $1/2\pi$ of the multiplication property and the weight π of each impulse give the factor $\frac{1}{2}$ on each copy. The frequency-shift property gives the same result term by term: $\frac{1}{2}e^{j\omega_c t}x(t) \leftrightarrow \frac{1}{2}X\big(j(\omega-\omega_c)\big)$ and $\frac{1}{2}e^{-j\omega_c t}x(t) \leftrightarrow \frac{1}{2}X\big(j(\omega+\omega_c)\big)$.

DOUBLE-SIDEBAND SUPPRESSED-CARRIER MODULATION

$$z(t) = x(t)\cos(\omega_c t) \;\longleftrightarrow\; Z(j\omega) = \tfrac{1}{2}X\big(j(\omega-\omega_c)\big) + \tfrac{1}{2}X\big(j(\omega+\omega_c)\big)$$

The spectrum is not moved to ω_c; it is **duplicated**, one copy at $+\omega_c$ and one at $-\omega_c$, each at half height. If $X(j0) = 1$ and the carrier is $\cos(10\pi t)$, the copy centred at 10π peaks at $Z(j10\pi) = \frac{1}{2}X(j0) = 0.5$. A description that says "the signal moves up to the carrier" loses the negative-frequency copy and the factor of one half in the same sentence.

EXAMPLE 5.20 — MODULATING A COSINE

GIVEN $x(t) = \cos(\pi t)$ and the carrier $\cos(4\pi t)$.

FIND The impulses of $Z(j\omega)$ for $z(t) = x(t)\cos(4\pi t)$, by two routes.

METHOD Route 1: turn the product into a sum with a trigonometric identity, then transform each cosine. Route 2: apply the modulation result to $X(j\omega)$.

SOLUTION Route 1. The product-to-sum identity $\cos\alpha\cos\beta = \frac{1}{2}\cos(\alpha-\beta) + \frac{1}{2}\cos(\alpha+\beta)$ with $\alpha = 4\pi t$ and $\beta = \pi t$ gives

$$z(t) = \tfrac{1}{2}\cos(3\pi t) + \tfrac{1}{2}\cos(5\pi t).$$

A cosine of unit amplitude transforms to two impulses of weight π, so each $\frac{1}{2}$ cos gives two impulses of weight $\frac{\pi}{2}$:

$$Z(j\omega) = \tfrac{\pi}{2}\big[\delta(\omega-3\pi) + \delta(\omega+3\pi) + \delta(\omega-5\pi) + \delta(\omega+5\pi)\big].$$

Route 2. $X(j\omega) = \pi\delta(\omega-\pi) + \pi\delta(\omega+\pi)$. The modulation result halves X and centres one copy at $+4\pi$ and one at -4π. The copy at $+4\pi$ has impulses of weight $\pi/2$ at $4\pi \pm \pi$, that is at 3π and 5π; the copy at -4π has them at -3π and -5π.

CHECK Both routes give four impulses of weight $\pi/2 = 1.570796$, at $\pm 3\pi$ and $\pm 5\pi$. Nothing sits at $\pm\omega_c = \pm 4\pi$ itself: only the sidebands are sent. That is what "suppressed carrier" records.

EXAMPLE 5.21 – MODULATING A BAND-LIMITED SIGNAL

GIVEN $x(t) = \dfrac{\sin(2\pi t)}{\pi t}$, so $X(j\omega) = 1$ on $|\omega| < 2\pi$ (Example 5.6), and $z(t) = x(t)\cos(4\pi t)$
.

FIND $Z(j\omega)$ and its band edges.

METHOD Two half-height copies of X, centred at $+4\pi$ and -4π. Find the edges of each copy.

SOLUTION The copy $\frac{1}{2}X\big(j(\omega-4\pi)\big)$ is 0.5 where $|\omega - 4\pi| < 2\pi$, that is on $2\pi < \omega < 6\pi$. The copy $\frac{1}{2}X\big(j(\omega+4\pi)\big)$ is 0.5 on $-6\pi < \omega < -2\pi$. The copies do not meet, so

$$Z(j\omega) = 0.5 \quad \text{on} \quad 2\pi \le |\omega| \le 6\pi, \qquad 0 \text{ elsewhere.}$$

CHECK Each copy is 4π wide, as wide as X, at half its height; together they occupy 8π of the axis. In time the carrier fills the envelope $\pm x(t)$. Multiplying z by the carrier again brings one pair of copies back to the origin, where they add to $\frac{1}{2}X$. A low-pass filter on $|\omega| < 2\pi$ then recovers x up to the factor $\frac{1}{2}$; this is the receiver described below.

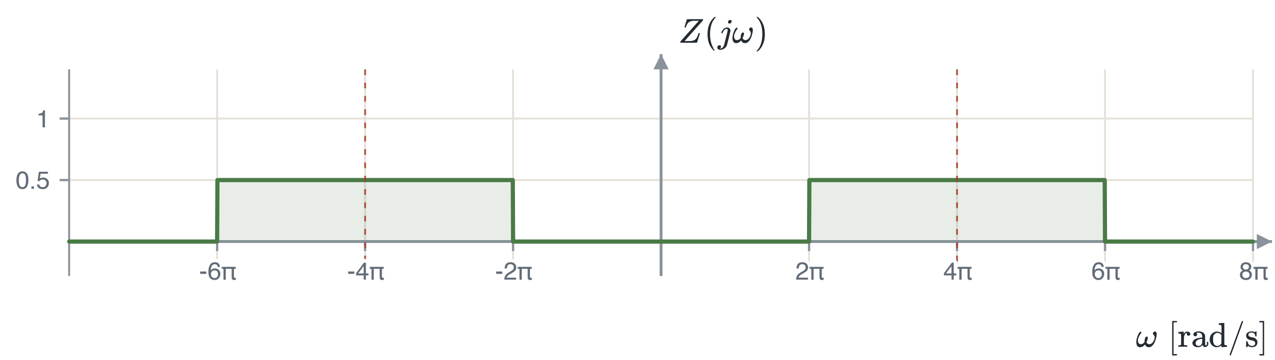

FIGURE 5.23 Two half-height copies on $2\pi \le |\omega| \le 6\pi$. The dashed lines mark $\pm\omega_c = \pm 4\pi$.

EXAMPLE 5.22 – COPIES THAT OVERLAP

GIVEN $X(j\omega) = 1$ on $\pi \le |\omega| \le 3\pi$ and 0 elsewhere, and $z(t) = x(t)\cos(2\pi t)$.

FIND $Z(j\omega)$, and $Z(j0)$ in particular.

METHOD Shift X up by 2π and down by 2π, halve both copies, and add them where they land on the same stretch.

SOLUTION The band of X has two parts, $\pi \le \omega \le 3\pi$ and $-3\pi \le \omega \le -\pi$. Shifted up by 2π they land on $3\pi \le \omega \le 5\pi$ and $-\pi \le \omega \le \pi$. Shifted down by 2π they land on $-\pi \le \omega \le \pi$ and $-5\pi \le \omega \le -3\pi$. Each copy has height $\frac{1}{2}$. On $|\omega| \le \pi$ both copies land, and their heights add to $\frac{1}{2} + \frac{1}{2} = 1$:

$$Z(j\omega) = \begin{cases} 1, & |\omega| \le \pi \\ 0.5, & 3\pi \le |\omega| \le 5\pi \\ 0, & \text{elsewhere.} \end{cases}$$

So $Z(j0) = 1$.

CHECK The total area is kept: X has area $2 \cdot 2\pi = 4\pi$, and Z has area $1 \cdot 2\pi + 0.5 \cdot 2\pi + 0.5 \cdot 2\pi = 4\pi$, as a sum of two half-height copies of X must have.

TWO DIFFERENT EVENTS

Copies appear whenever a signal is multiplied by a carrier. That happens at every carrier frequency, and it loses nothing. **Copies overlap** only when the carrier is low enough for the shifted bands to reach each other, as in Example 5.22. Overlap is what destroys information: once two copies have been added, there is no way to tell what each contributed. Chapter 7 asks the same question about the copies that sampling produces.

Synchronous demodulation

A modulated signal $z(t) = x(t)\cos(\omega_c t)$ arrives at a receiver. The message is band-limited, $X(j\omega) = 0$ for $|\omega| > W$, with $W < \omega_c$. The receiver multiplies by the same carrier again. The identity $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$ splits the product into the message and a new modulated signal:

$$y(t) = z(t)\cos(\omega_c t) = x(t)\cos^2(\omega_c t) = \tfrac{1}{2}x(t) + \tfrac{1}{2}x(t)\cos(2\omega_c t).$$

Transform term by term. The first term gives $\frac{1}{2}X(j\omega)$. The second is the modulation result with carrier $2\omega_c$, times $\frac{1}{2}$: two copies of height $\frac{1}{4}$ centred at $\pm 2\omega_c$:

$$Y(j\omega) = \tfrac{1}{2}X(j\omega) + \tfrac{1}{4}X\big(j(\omega - 2\omega_c)\big) + \tfrac{1}{4}X\big(j(\omega + 2\omega_c)\big).$$

The copy at $+2\omega_c$ occupies $2\omega_c - W \le \omega \le 2\omega_c + W$, so it begins at $2\omega_c - W$. A low-pass filter of gain 2 whose cutoff lies between W and $2\omega_c - W$ keeps $\frac{1}{2}X$, doubles it, and removes both copies. Its output is $x(t)$. The interval for the cutoff is not empty because $W < 2\omega_c - W$ is the same as $W < \omega_c$. With $W = 3\pi$ and $\omega_c = 5\pi$ rad/s, for example, the cutoff may be as high as $2\omega_c - W = 10\pi - 3\pi = 7\pi$.

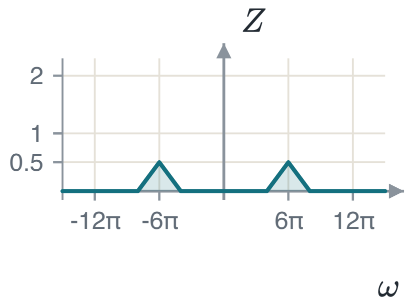

FIGURE 5.24 $Z(j\omega)$ at the receiver.

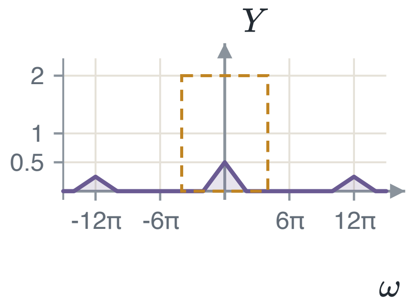

FIGURE 5.25 $Y(j\omega)$ and the filter (dashed box).

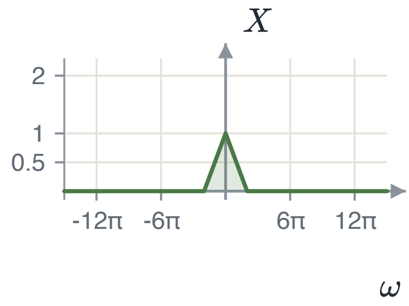

FIGURE 5.26 The filter output: $X(j\omega)$.

In the figure $W = 2\pi$ and $\omega_c = 6\pi$. The copies of Y lie on $10\pi \le |\omega| \le 14\pi$, far outside a filter of gain 2 on $|\omega| < 4\pi$.

The receiver must use the carrier with the right phase, which is why the scheme is called **synchronous**. Suppose it multiplies by $\cos(\omega_c t + \phi)$. The identity $\cos\alpha \cos\beta = \frac{1}{2}\cos(\alpha - \beta) + \frac{1}{2}\cos(\alpha + \beta)$ with $\alpha = \omega_c t + \phi$ and $\beta = \omega_c t$ gives $\cos(\omega_c t)\cos(\omega_c t + \phi) = \frac{1}{2}\cos\phi + \frac{1}{2}\cos(2\omega_c t + \phi)$. The filter keeps and doubles the first term only, and returns $x(t)\cos\phi$. At $\phi = \pi/2$ nothing is left.

A band-pass filter with a tunable centre

A fixed ideal low-pass filter H_{lp}, with gain 1 for $|\omega| < \omega_0$ and 0 beyond, can pass a band around any chosen frequency ω_c. Multiply the input by $e^{-j\omega_c t}$, filter, and multiply by $e^{j\omega_c t}$. Only the oscillator frequency ω_c is turned; the filter never changes:

$$x(t) \xrightarrow{\times\, e^{-j\omega_c t}} v_1(t) \longrightarrow \boxed{H_{\text{lp}}} \longrightarrow v_2(t) \xrightarrow{\times\, e^{j\omega_c t}} y(t)$$

Follow the spectrum through the three steps. Each multiplication by $e^{\pm j\omega_c t}$ is a frequency shift. The frequency-shift property with the shift $-\omega_c$ replaces ω by $\omega + \omega_c$. The filter multiplies. The last shift replaces ω by $\omega - \omega_c$ in everything before it:

$$\begin{aligned} V_1(j\omega) &= X\big(j(\omega + \omega_c)\big), \\ V_2(j\omega) &= H_{\text{lp}}(j\omega)\, X\big(j(\omega + \omega_c)\big), \\ Y(j\omega) &= V_2\big(j(\omega - \omega_c)\big) = H_{\text{lp}}\big(j(\omega - \omega_c)\big)\, X(j\omega). \end{aligned}$$

$H_{\text{lp}}\big(j(\omega - \omega_c)\big) = 1$ exactly when $|\omega - \omega_c| < \omega_0$. So $Y = X$ on $\omega_c - \omega_0 < \omega < \omega_c + \omega_0$ and $Y = 0$ elsewhere: a band-pass filter of width $2\omega_0$ centred at ω_c. With $\omega_c = 12$ and $\omega_0 = 3$ rad/s, only $9 < \omega < 15$ reaches the output; with $\omega_c = 8$ and $\omega_0 = 1$ rad/s, only $7 < \omega < 9$.

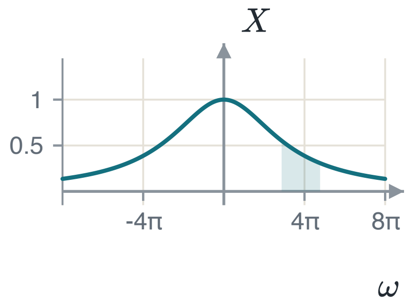

FIGURE 5.27 $X(j\omega)$; the band near ω_c is shaded.

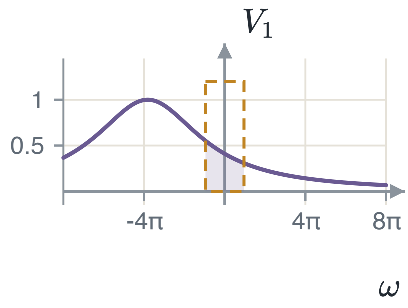

FIGURE 5.28 Shifted down by ω_c; the dashed box is H_{lp}.

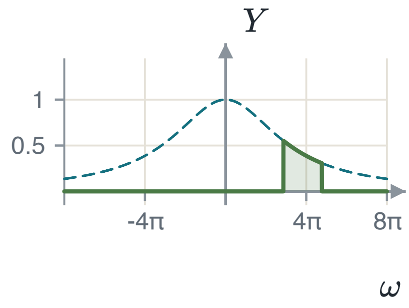

FIGURE 5.29 Shifted back: $Y(j\omega)$ on $9 < \omega < 15$.

The band near $-\omega_c$ is not passed. So Y is not conjugate-symmetric, and $y(t)$ is complex even when $x(t)$ is real (Section 5.4).

EXAMPLE 5.23 — THE SQUARE OF A SINC

GIVEN $x(t) = \dfrac{\sin(2\pi t)}{\pi t}$, so $X(j\omega) = 1$ on $|\omega| < 2\pi$, and $z(t) = x^2(t)$.

FIND $Z(j\omega)$ and $Z(j0)$.

METHOD Multiplication in time is convolution in frequency, with $1/2\pi$. Convolve the band with itself by sliding one copy across the other and measuring the overlap.

SOLUTION Two rectangles of height A and half-width ω_0 convolve to a triangle. At zero shift the overlap is the full $2\omega_0$ wide, which gives the apex $A \cdot A \cdot 2\omega_0 = 2A^2\omega_0$. The overlap shrinks linearly to nothing at shift $\pm 2\omega_0$, so the triangle lives on $|\omega| \le 2\omega_0$. Here $A = 1$ and $\omega_0 = 2\pi$, so $X * X$ is a triangle of apex $2 \cdot 1 \cdot 2\pi = 4\pi$ on $|\omega| \le 4\pi$. The $1/2\pi$ of the multiplication property scales the apex to $4\pi/2\pi = 2$:

$$Z(j\omega) = \frac{1}{2\pi}(X * X)(j\omega) = 2\left(1 - \frac{|\omega|}{4\pi}\right), \qquad |\omega| \le 4\pi,$$

and 0 beyond. So $Z(j0) = 2$. The spectrum of the square is twice as wide as X.

CHECK In time, $x(0) = 2\pi/\pi = 2$ (Example 5.6 with $W = 2\pi$), so $z(0) = x(0)^2 = 4$. Through the transform, $z(0) = \frac{1}{2\pi}\int Z\,\mathrm{d}\omega$, and the triangle has area $\frac{1}{2} \cdot 8\pi \cdot 2 = 8\pi$, so $z(0) = 8\pi/2\pi = 4$. The two agree.

EXAMPLE 5.24 — THE PRODUCT OF TWO DIFFERENT BANDS

GIVEN $x_1(t) = \dfrac{\sin(2\pi t)}{\pi t}$ and $x_2(t) = \dfrac{\sin(4\pi t)}{\pi t}$, so $X_1 = 1$ on $|\omega| < 2\pi$ and $X_2 = 1$ on $|\omega| < 4\pi$.

FIND The transform Z of $z(t) = x_1(t)x_2(t)$, and the peak $z(0)$.

METHOD As in Example 5.23: convolve the bands, then divide by 2π. Track the overlap as the narrow band slides across the wide one.

SOLUTION For shifts $|\omega| \le 4\pi - 2\pi = 2\pi$, the narrow band lies wholly inside the wide one, so the overlap stays at its full width 4π. The convolution is flat there at 4π, and $Z = 4\pi/2\pi = 2$. Beyond that the overlap shrinks linearly and reaches zero when the bands just touch, at the sum of the half-widths, $|\omega| = 2\pi + 4\pi = 6\pi$. So Z is a trapezoid: 2 on $|\omega| \leq 2\pi$, falling linearly to 0 at $|\omega| = 6\pi$. The flat top has half-width equal to the difference of the half-widths, and it shrinks to a point when they are equal, which gives the triangle of Example 5.23.

CHECK The peak in time is the product of the two peaks, $z(0) = x_1(0)x_2(0) = 2 \times 4 = 8$. The trapezoid has parallel sides 4π and 12π and height 2, so its area is $\frac{1}{2}(4\pi + 12\pi) \cdot 2 = 16\pi$, and $z(0) = 16\pi/2\pi = 8$. The two agree.

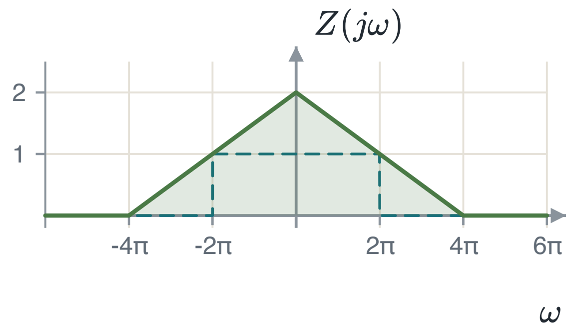

FIGURE 5.30 Example 5.23: the triangle of apex 2 on $|\omega| \leq 4\pi$; dashed, the band X.

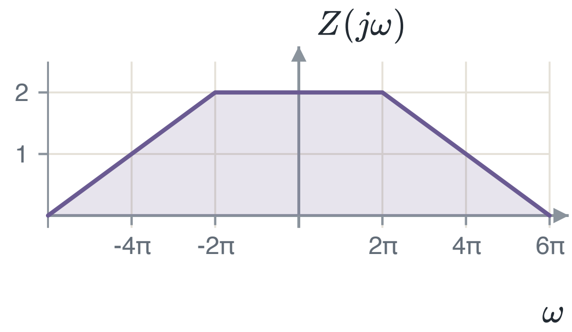

FIGURE 5.31 Example 5.24: the trapezoid, flat on $|\omega| \leq 2\pi$ and zero beyond 6π.

5.6 Systems from a differential equation

The Fourier transform converts a linear differential equation with constant coefficients into an algebraic equation. Start from

$$\sum_{k=0}^{N} a_k \frac{\mathrm{d}^k y(t)}{\mathrm{d}t^k} = \sum_{k=0}^{M} b_k \frac{\mathrm{d}^k x(t)}{\mathrm{d}t^k}.$$

Take the transform of both sides. Linearity moves the transform inside each sum and past each constant coefficient, and the differentiation property replaces each derivative $\mathrm{d}^k/\mathrm{d}t^k$ by the factor $(j\omega)^k$:

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega).$$

Both sides now contain a plain product. Factor $Y(j\omega)$ out of the left sum and $X(j\omega)$ out of the right sum, and divide. The ratio Y/X is the frequency response, because for an LTI system $Y = XH$:

FREQUENCY RESPONSE FROM THE COEFFICIENTS

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

The coefficients of the differential equation form two polynomials in $j\omega$, and their ratio is the frequency response. $H(j\omega)$ exists when h is absolutely integrable, $\int |h(t)|\, \mathrm{d}t < \infty$. For an LTI system that is bounded-input bounded-output stability, so an unstable system has no frequency response to plot. The route of this section: transform the equation, read $H(j\omega)$, expand it into simple fractions, and invert each term with the table of pairs.

During the algebra it helps to write $s = j\omega$. Here s is only a name for $j\omega$ that shortens the partial-fraction algebra; every expression is a function of the real frequency ω, and $j\omega$ is put back at the end. The two inverse pairs needed are $1/(s+a) = 1/(a+j\omega) \leftrightarrow e^{-at}u(t)$ from Example 5.3 and, below, its repeated form.

EXAMPLE 5.25 – SIMPLE POLES

GIVEN $\dfrac{\mathrm{d}^2y}{\mathrm{d}t^2} + 4\dfrac{\mathrm{d}y}{\mathrm{d}t} + 3y = \dfrac{\mathrm{d}x}{\mathrm{d}t} + 2x$, at rest.

FIND $H(j\omega)$ and $h(t)$.

METHOD The differential equation has constant coefficients, so the differentiation property turns it into algebra. Form the frequency-response ratio, factor it, expand it into simple fractions, and invert each term.

SOLUTION Take the transform of both sides. Each derivative becomes a power of $j\omega$, and linearity keeps the coefficients:

$$(j\omega)^2 Y(j\omega) + 4(j\omega)Y(j\omega) + 3Y(j\omega) = (j\omega)X(j\omega) + 2X(j\omega).$$

Write $s = j\omega$, factor out Y on the left and X on the right, and divide:

$$H = \frac{Y}{X} = \frac{s+2}{s^2+4s+3}.$$

Factor the denominator. Its roots are the values of s where $s^2 + 4s + 3 = 0$, that is $s = -1$ and $s = -3$, so $s^2 + 4s + 3 = (s+1)(s+3)$. The roots are called the **poles**. Expand into simple fractions:

$$\frac{s+2}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3}, \qquad s+2 = A(s+3) + B(s+1).$$

Set $s = -1$ so the B term vanishes: $-1+2 = A(-1+3)$, so $1 = 2A$ and $A = \frac{1}{2}$. Set $s = -3$ so the A term vanishes: $-3+2 = B(-3+1)$, so $-1 = -2B$ and $B = \frac{1}{2}$. This is the **cover-up rule**: cover one factor, and evaluate the rest at the root of that factor, $A = \frac{s+2}{s+3}\big|_{s=-1}$ and $B = \frac{s+2}{s+1}\big|_{s=-3}$. Hence

$$H(j\omega) = \frac{1/2}{1+j\omega} + \frac{1/2}{3+j\omega},$$

and each fraction is the pair of Example 5.3, so

$$h(t) = \left[\tfrac{1}{2}e^{-t} + \tfrac{1}{2}e^{-3t}\right]u(t).$$

CHECK At $\omega = 0$, $H(j0) = 2/3$ from the ratio. The transform at $\omega = 0$ is the area, and $\int_0^\infty h\,\mathrm{d}t = \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$, which agrees. Also $h(0^+) = \frac{1}{2} + \frac{1}{2} = 1$. The poles $s = -1$ and $s = -3$ are negative, so both exponentials decay, h is absolutely integrable, the system is stable, and H exists.

The frequency response of the example

The modulus of a quotient of products is the quotient of the products of the moduli. The angle is the numerator angle minus the denominator angles, and each factor $c + j\omega$ with $c > 0$ has angle $\tan^{-1}(\omega/c)$:

$$|H(j\omega)| = \frac{|2 + j\omega|}{|1 + j\omega|\,|3 + j\omega|} = \frac{\sqrt{4 + \omega^2}}{\sqrt{1 + \omega^2}\,\sqrt{9 + \omega^2}},$$
$$\angle H(j\omega) = \tan^{-1}\frac{\omega}{2} - \tan^{-1}\omega - \tan^{-1}\frac{\omega}{3}.$$

At $\omega = 0$ the system passes $2/3 = 0.667$ of a constant. For large ω the numerator has one factor and the denominator two, so $|H| \approx \omega/\omega^2 = 1/\omega$; at $\omega = 100$ rad/s, $|H| = 0.009997 \approx 0.01$. Each arctangent tends to $\pi/2$, so the phase tends to $\pi/2 - \pi/2 - \pi/2 = -\pi/2$. The magnitude is even and the phase is odd, as a real h requires.

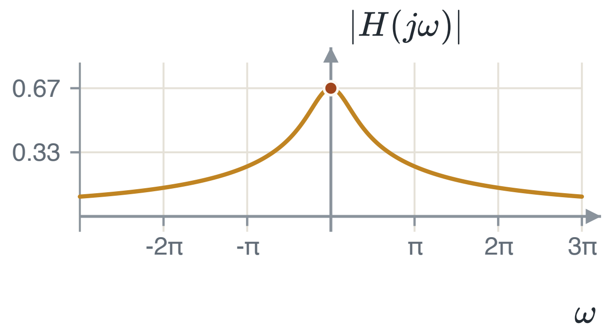

FIGURE 5.32 The magnitude, $2/3$ at $\omega = 0$.

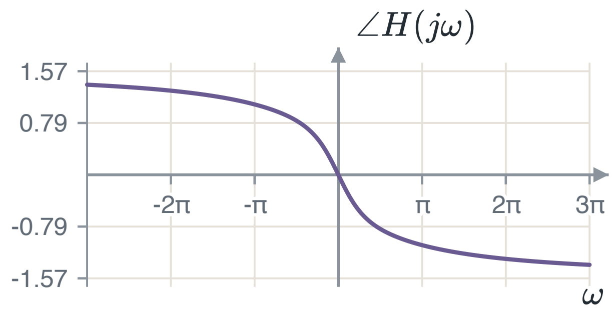

FIGURE 5.33 The phase in rad, odd in ω.

Repeated poles

The cover-up rule assumes each factor of the denominator appears once. When a factor $(s - \lambda)$ appears m times, one term is needed for each power:

$$F(s) = \frac{N(s)}{(s - \lambda)^m Q(s)} = \frac{c_m}{(s - \lambda)^m} + \cdots + \frac{c_1}{s - \lambda} + (\text{terms from } Q).$$

Covering up fails here. Multiply by $(s - \lambda)$ only, and $m - 1$ factors survive below, so setting $s = \lambda$ divides by zero. Multiply by the full $(s - \lambda)^m$ instead: then $(s - \lambda)^m F(s) = c_m + c_{m-1}(s - \lambda) + \cdots + c_1(s - \lambda)^{m-1} + (s - \lambda)^m(\text{terms from } Q)$. Setting $s = \lambda$ removes every term except c_m. To reach the next coefficient, differentiate once before setting $s = \lambda$: the constant c_m disappears, the term $c_{m-1}(s - \lambda)$ leaves c_{m-1}, and every other term still carries a factor $(s - \lambda)$ and vanishes. Each further derivative peels off one more coefficient, with a factorial from differentiating the power.

REPEATED-POLE RULE

$$c_{m-k} = \frac{1}{k!}\,\frac{\mathrm{d}^k}{\mathrm{d}s^k}\Big[(s - \lambda)^m F(s)\Big]\bigg|_{s=\lambda}, \qquad k = 0, 1, \ldots, m - 1$$

The case $k = 0$ is the cover-up rule and gives c_m.

The inverse pairs follow from the differentiation-in-frequency property of Section 5.4. That property gave $te^{-at}u(t) \leftrightarrow 1/(a+j\omega)^2$. Suppose $\frac{t^{n-1}}{(n-1)!}e^{-at}u(t) \leftrightarrow (a+j\omega)^{-n}$ for some $n \geq 1$. Multiply the signal by t/n; the property multiplies the transform by j/n and differentiates it:

$$\frac{t^n}{n!}e^{-at}u(t) \longleftrightarrow \frac{j}{n}\frac{\mathrm{d}}{\mathrm{d}\omega}(a+j\omega)^{-n} = \frac{j}{n}\cdot(-n)\,j\,(a+j\omega)^{-n-1} = \frac{1}{(a+j\omega)^{n+1}}.$$

So by induction $\frac{t^{n-1}}{(n-1)!}e^{-at}u(t) \leftrightarrow 1/(a+j\omega)^n$ for every $n \geq 1$, with $a > 0$. A repeated pole always brings a power of t into the time domain. The double pole gives $te^{-at}u(t)$, which starts at 0 and peaks at $t = 1/a$; for example $1/(j\omega+2)^2 \leftrightarrow te^{-2t}u(t)$.

EXAMPLE 5.26 – A REPEATED POLE, AND THE CHECK THAT CATCHES A SIGN

GIVEN The system of Example 5.25, $H(j\omega) = \dfrac{j\omega+2}{(j\omega+1)(j\omega+3)}$, with input $x(t) = e^{-t}u(t)$.

FIND $y(t)$.

METHOD The input passes through an LTI system, so calculate $Y = XH$. Write $s = j\omega$, expand Y into partial fractions with the repeated-pole rule, and invert each term.

SOLUTION Example 5.3 gives $X = 1/(s+1)$. Multiply:

$$Y = \frac{1}{s+1}\cdot\frac{s+2}{(s+1)(s+3)} = \frac{s+2}{(s+1)^2(s+3)}.$$

The input pole coincides with a system pole, so $s=-1$ is now a double pole. One term is needed for each power of $(s+1)$, so Y splits into three terms:

$$Y = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+3}.$$

For B, multiply by $(s+1)^2$ and set $s=-1$:

$$B = \Big[(s+1)^2 Y\Big]_{s=-1} = \frac{s+2}{s+3}\bigg|_{s=-1} = \frac{-1+2}{-1+3} = \frac{1}{2}.$$

For A, differentiate the same product once before setting $s=-1$. By the quotient rule, $\frac{\mathrm{d}}{\mathrm{d}s}\frac{s+2}{s+3} = \frac{(s+3)\cdot 1-(s+2)\cdot 1}{(s+3)^2} = \frac{1}{(s+3)^2}$, so

$$A = \frac{\mathrm{d}}{\mathrm{d}s}\Big[(s+1)^2 Y\Big]_{s=-1} = \frac{1}{(s+3)^2}\bigg|_{s=-1} = \frac{1}{2^2} = \frac{1}{4}.$$

For C, the factor $(s+3)$ is simple, so cover it up and set $s=-3$:

$$C = \Big[(s+3)Y\Big]_{s=-3} = \frac{s+2}{(s+1)^2}\bigg|_{s=-3} = \frac{-3+2}{(-3+1)^2} = \frac{-1}{4} = -\frac{1}{4}.$$

C is negative, and the minus is most easily lost on the line where the three fractions are written out together. Invert term by term with $1/(s+a) \leftrightarrow e^{-at}u(t)$ and $1/(s+a)^2 \leftrightarrow te^{-at}u(t)$:

$$y(t) = \left[\tfrac{1}{4}e^{-t} + \tfrac{1}{2}t\,e^{-t} - \tfrac{1}{4}e^{-3t}\right]u(t).$$

CHECK

$y = x * h$ with x and h causal, so $y(0) = 0$. The answer gives $y(0) = \frac{1}{4} + 0 - \frac{1}{4} = 0$. Assembling C as $+\frac{1}{4}$ instead gives $y(0) = \frac{1}{4} + 0 + \frac{1}{4} = \frac{1}{2}$, which no convolution of two causal signals can give. Comparing curves elsewhere decides little: at $t = 2$ the two candidates are 0.1685 and 0.1698, which agree only to two decimals, while at $t = 0$ they differ by 0.5. Convolving x with h directly,

$$y(t) = \int_0^t e^{-(t-\tau)} \left[\tfrac{1}{2}e^{-\tau} + \tfrac{1}{2}e^{-3\tau}\right] \mathrm{d}\tau = e^{-t}\left[\tfrac{1}{2}t + \tfrac{1}{4}\left(1 - e^{-2t}\right)\right],$$

returns the same three terms with the same signs. The middle step uses $\int_0^t \frac{1}{2}e^{-2\tau}\,\mathrm{d}\tau = \frac{1}{4}(1 - e^{-2t})$.

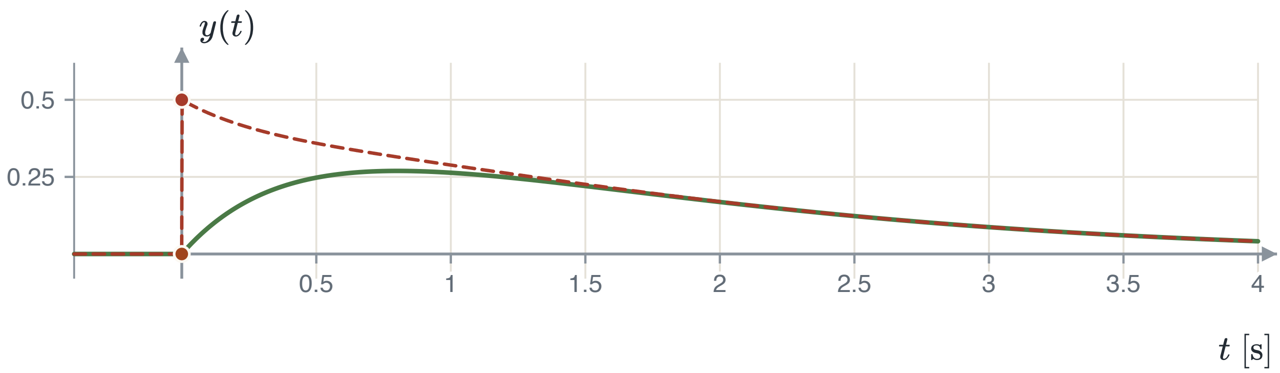

FIGURE 5.34 The output with $C = -\frac{1}{4}$ (solid) and, dashed, the version with the sign lost. They differ by 0.5 at the origin and by about 0.001 at $t = 2$.

5.7 Summary

Properties

TABLE 5.1 Properties of the continuous-time Fourier transform.

PROPERTY	STATEMENT
Linearity	$ax_1(t) + bx_2(t) \;\leftrightarrow\; aX_1(j\omega) + bX_2(j\omega)$
Time shift	$x(t - t_0) \;\leftrightarrow\; e^{-j\omega t_0}X(j\omega)$
Frequency shift	$e^{j\omega_0 t}x(t) \;\leftrightarrow\; X\big(j(\omega - \omega_0)\big)$
Conjugation	$x^*(t) \;\leftrightarrow\; X^*(-j\omega)$
Time reversal	$x(-t) \;\leftrightarrow\; X(-j\omega)$
Time scaling	$x(at) \;\leftrightarrow\; \frac{1}{\lvert a\rvert}X(j\omega/a)$
Convolution	$x(t) * h(t) \;\leftrightarrow\; X(j\omega)H(j\omega)$
Multiplication	$x(t)y(t) \;\leftrightarrow\; \frac{1}{2\pi}X(j\omega) * Y(j\omega)$
Duality	$X(t) \;\leftrightarrow\; 2\pi x(-\omega)$
Differentiation in time	$\mathrm{d}^n x/\mathrm{d}t^n \;\leftrightarrow\; (j\omega)^n X(j\omega)$
Integration	$\int_{-\infty}^{t} x(\tau)\,\mathrm{d}\tau \;\leftrightarrow\; \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
Differentiation in frequency	$t\,x(t) \;\leftrightarrow\; j\,\mathrm{d}X(j\omega)/\mathrm{d}\omega$
Real signal	$X(-j\omega) = X^*(j\omega)$: $\lvert X\rvert$ even, $\angle X$ odd
Real and even	$X(j\omega)$ real and even
Real and odd	$X(j\omega)$ purely imaginary and odd
Even and odd parts	$\mathcal{E}\mathrm{v}\{x\} \leftrightarrow \mathrm{Re}\{X\}$, $\mathcal{O}\mathrm{dd}\{x\} \leftrightarrow j\,\mathrm{Im}\{X\}$
Parseval	$\int \lvert x(t)\rvert^2\,\mathrm{d}t = \frac{1}{2\pi}\int \lvert X(j\omega)\rvert^2\,\mathrm{d}\omega$

Two rows carry a condition. Integration keeps $\pi X(0)\delta(\omega)$ whenever the signal has non-zero area. Scaling carries $1/\lvert a\rvert$, with the modulus, so a reversal is counted once. Check every answer at $\omega = 0$ against the area.

Transform pairs

TABLE 5.2 Continuous-time Fourier transform pairs.

SIGNAL	PAIR
Impulse	$\delta(t) \leftrightarrow 1$
Shifted impulse	$\delta(t-t_0) \leftrightarrow e^{-j\omega t_0}$
Unit step	$u(t) \leftrightarrow \frac{1}{j\omega} + \pi\delta(\omega)$
One-sided exponential	$e^{-at}u(t) \leftrightarrow \frac{1}{a+j\omega}$, $a>0$
Repeated pole	$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t) \leftrightarrow \frac{1}{(a+j\omega)^n}$, $a>0$
Two-sided exponential	$e^{-a\|t\|} \leftrightarrow \frac{2a}{a^2+\omega^2}$, $a>0$
Rectangular pulse	1 on $\|t\|<T_1 \leftrightarrow \frac{2\sin(\omega T_1)}{\omega} = 2T_1\,\mathrm{sinc}(\omega T_1)$
Ideal low-pass band	$\frac{\sin(Wt)}{\pi t} \leftrightarrow 1$ on $\|\omega\|<W$
Constant	$1 \leftrightarrow 2\pi\delta(\omega)$
Complex exponential	$e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega-\omega_0)$
Cosine	$\cos(\omega_0 t) \leftrightarrow \pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$
Sine	$\sin(\omega_0 t) \leftrightarrow \frac{\pi}{j}\delta(\omega-\omega_0)-\frac{\pi}{j}\delta(\omega+\omega_0)$
Periodic signal	$\sum_k a_k e^{jk\omega_0 t} \leftrightarrow \sum_k 2\pi a_k\delta(\omega-k\omega_0)$
Square wave	1 on $\|t\|<T_1$ in each period $T \leftrightarrow \sum_{k\neq 0}\frac{2\sin(k\omega_0 T_1)}{k}\delta(\omega-k\omega_0)+\frac{4\pi T_1}{T}\delta(\omega)$
Impulse train	$\sum_k \delta(t-kT) \leftrightarrow \frac{2\pi}{T}\sum_k \delta\left(\omega-\frac{2\pi k}{T}\right)$

Every sinc in the table is unnormalised, $\mathrm{sinc}(\theta)=\sin\theta/\theta$; a table in the normalised convention divides the argument by π. The exponential pairs hold only for $a>0$.

What to carry forward

- Let the period of a pulse train grow: Ta_k samples one envelope, and in the limit the envelope is $X(j\omega)$.
- Analysis integrates over t and returns $X(j\omega)$; synthesis integrates over ω, carries $1/2\pi$, and returns $x(t)$.
- Finite energy **or** the Dirichlet conditions is enough; neither is necessary. At a jump the synthesis integral gives the midpoint, and a cut-off band leaves an overshoot of about 0.09 that does not shrink.
- The sinc convention here is $\mathrm{sinc}(\theta)=\sin\theta/\theta$, and it is stated at every point of use.
- Narrowing a signal in time widens its spectrum. Finite duration forbids band limitation; infinite duration guarantees nothing.
- A periodic signal transforms to impulses of weight $2\pi a_k$ at the harmonics, not to the coefficients themselves.

- A delay multiplies by $e^{-j\omega t_0}$: the magnitude stays and the phase gains $-\omega t_0$. A real signal has an even magnitude and an odd phase.
- Convolution in time is multiplication in frequency; multiplication in time is convolution in frequency, with $1/2\pi$. A carrier makes two copies at half height. Copies always appear; overlap is a separate event and is what loses information.
- Parseval: $\int |x|^2 \,\mathrm{d}t = \frac{1}{2\pi}\int |X|^2 \,\mathrm{d}\omega$, with $R = 1\,\Omega$.
- A differential equation gives $H(j\omega)$ as a ratio of polynomials in $j\omega$; partial fractions and the table give $h(t)$. A repeated pole needs the derivative rule, and a causal convolution must start at zero.

Everything here was continuous time. A computer sees a sequence, not a signal, and the same question has to be asked again for $x[n]$. One thing changes, and it changes everything: $e^{-j(\omega+2\pi)n} = e^{-j\omega n}$ for every integer n, so a discrete-time spectrum repeats with period 2π. That is Chapter 6.

Exercises

5.1 A pulse train is 1 on $|t| < 1$ in each period $T_0 = 6$ s. Write $T_0 a_k$ as a sample of the pulse envelope and find a_0 and a_1.

Answer: The envelope is the transform of one pulse, $E(\omega) = 2\sin\omega/\omega$, and $T_0 a_k = E(k\omega_0)$ with $\omega_0 = 2\pi/6 = \pi/3$ rad/s. At $k = 0$ the envelope is its limit 2, so $a_0 = 2/6 = 1/3$. At $k = 1$, $a_1 = \frac{1}{6}\cdot\frac{2\sin(\pi/3)}{\pi/3} = \frac{1}{6}\cdot\frac{2(0.8660)(3)}{\pi} = 0.2757$.

5.2 Find the transform of $x(t) = e^{-3|t|}$ and give its values at $\omega = 0$ and $\omega = 3$ rad/s.

Answer: The two-sided pair with $a = 3$ gives $X(j\omega) = \frac{6}{9+\omega^2}$. So $X(j0) = 6/9 = 2/3$ and $X(j3) = 6/18 = 1/3$: at $\omega = a$ the spectrum is half its peak.

5.3 A pulse is 1 on $|t| < 2$. Write its transform with the unnormalised sinc, $\operatorname{sinc}\theta = \sin\theta/\theta$, and find $X(j0)$ and the first zero with $\omega > 0$.

Answer: With $T_1 = 2$, $X(j\omega) = \frac{2\sin(2\omega)}{\omega} = 4\operatorname{sinc}(2\omega)$. $X(j0) = 2T_1 = 4$, the area of the pulse. The first zero is at $2\omega = \pi$, so $\omega = \pi/2$ rad/s.

5.4 Find the transform of $x(t) = 2 + \cos(4t) - \sin(4t)$. Give the weight of each impulse and check that the two weights at ± 4 are conjugates.

Answer: Term by term: $2 \to 4\pi\delta(\omega)$, $\cos(4t) \to \pi\delta(\omega-4) + \pi\delta(\omega+4)$ and $-\sin(4t) \to -\frac{\pi}{j}\delta(\omega-4) + \frac{\pi}{j}\delta(\omega+4)$. With $1/j = -j$, the weight at $\omega = 4$ is $\pi + j\pi = \pi(1+j)$ and at $\omega = -4$ it is $\pi - j\pi = \pi(1-j)$. They are conjugates, as a real signal requires. The weight at $\omega = 0$ is 4π.

5.5 Find $|X(j2)|$ and $\angle X(j2)$ for $x(t) = e^{-2(t-3)}u(t-3)$.

Answer: The signal is $e^{-2t}u(t)$ delayed by $t_0 = 3$ s, so $X(j\omega) = \frac{e^{-j3\omega}}{2+j\omega}$. At $\omega = 2$: $|X(j2)| = 1/|2+j2| = 1/\sqrt{8} = 0.3536$, since the delay factor has magnitude 1. The phase is $-3(2) - \arctan(2/2) = -6 - \pi/4 = -6.785$ rad.

5.6 Use Parseval's relation to find the energy of $x(t) = \frac{\sin(4t)}{\pi t}$, with $R = 1\,\Omega$.

Answer: The ideal low-pass pair with $W = 4$ gives $X(j\omega) = 1$ on $|\omega| < 4$ and 0 outside. So $E = \frac{1}{2\pi}\int_{-4}^{4} 1^2 \,\mathrm{d}\omega = \frac{8}{2\pi} = \frac{4}{\pi} = 1.273$ J.

5.7 $X(j\omega)$ is zero for $|\omega| > 3$ rad/s and $X(j0) = 4$. Describe the spectrum of $x(t)\cos(10t)$, give the peak height of each copy, and find the smallest carrier frequency for which the copies do not overlap.

Answer: Multiplication by $\cos(10t)$ gives $\frac{1}{2}X(j(\omega-10)) + \frac{1}{2}X(j(\omega+10))$: two copies, on $7 \le \omega \le 13$ and $-13 \le \omega \le -7$, each peaking at $\frac{1}{2}\cdot 4 = 2$. With carrier ω_c the copies occupy $[\omega_c - 3, \omega_c + 3]$ and $[-\omega_c - 3, -\omega_c + 3]$. They stay apart when $\omega_c - 3 \ge -\omega_c + 3$, that is, $\omega_c \ge 3$ rad/s.

5.8 A system obeys $\frac{\mathrm{d}y}{\mathrm{d}t} + 4y(t) = 2x(t)$. Find $H(j\omega)$ and $h(t)$, then the output for $x(t) = e^{-2t}u(t)$.

Answer: Take the transform of both sides: $(j\omega + 4)Y = 2X$, so $H(j\omega) = \frac{2}{4+j\omega}$ and $h(t) = 2e^{-4t}u(t)$. With $X = \frac{1}{2+j\omega}$, $Y = \frac{2}{(2+j\omega)(4+j\omega)} = \frac{1}{2+j\omega} - \frac{1}{4+j\omega}$, from partial fractions with $A = 2/(4-2) = 1$ and $B = 2/(2-4) = -1$. So $y(t) = \left(e^{-2t} - e^{-4t}\right)u(t)$. Check: $y(0) = 0$, as a causal convolution requires.

CHAPTER

6

The discrete-time Fourier transform

The discrete-time Fourier transform describes the frequency content of a sequence that is not periodic. We derive it in three steps: replicate a finite sequence to make it periodic, apply the series from Chapter 4, and let the period grow without bound. The result is a continuous function of frequency that repeats every 2π. Every figure shows more than one period so that this property stays visible. The chapter then works out the standard pairs, the transforms of periodic sequences and every property with its proof. It ends with filters, windows and difference equations, and with the discrete Fourier transform that a computer evaluates.

6.1 Building the transform

Let $x[n]$ have finite support: $x[n] = 0$ for $|n| > N_1$. The discrete-time Fourier series applies to periodic sequences only, so build a periodic sequence out of x by laying the same finite sequence down again every N samples.

PERIODIC REPLICATION

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n - rN], \qquad \tilde{x}[n+N] = \tilde{x}[n]$$

Periodic replication places a copy of the complete sequence every N samples. It does not insert samples within a copy.

⚠ THE CONDITION THE CONSTRUCTION NEEDS

The copies must not overlap, so the period must be longer than the support: $N > 2N_1$. Only then is $\tilde{x}[n] = x[n]$ for $-N_1 \leq n \leq N_1$, a band of $2N_1 + 1$ samples centred on the origin. With $N \leq 2N_1$ the tails of neighbouring copies add and no period of $\tilde{x}$ equals x any more. For a sequence that is zero outside $|n| \leq 3$, the copies stay apart only for $N > 6$: $N = 9$ works, while $N = 4$ and $N = 6$ do not.

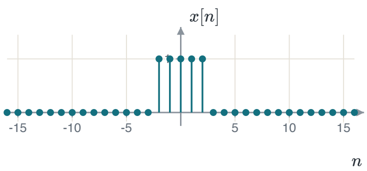

FIGURE 6.1 The finite-support sequence, $N_1 = 2$.

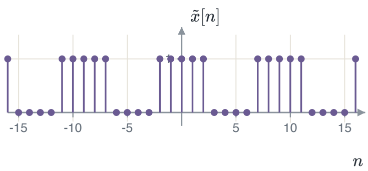

FIGURE 6.2 Replication with $N = 9 > 2N_1$: the copies stand clear.

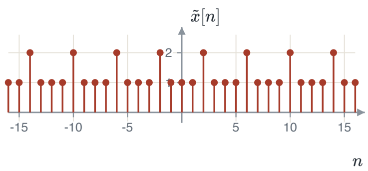

FIGURE 6.3 Replication with $N = 4 \leq 2N_1$: the copies add.

Let N grow while x stays fixed. The copies move outward, and at every fixed n the value $\tilde{x}[n]$ becomes $x[n]$ once N is large enough. So $\tilde{x}[n] \to x[n]$ at every n as $N \to \infty$.

The coefficients are samples of one function

DISCRETE-TIME FOURIER SERIES

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n}, \qquad \omega_0 = \frac{2\pi}{N}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} \tilde{x}[n]\, e^{-jk\omega_0 n}$$

The analysis exponent is negative. Synthesis builds the sequence up out of exponentials and carries $e^{+jk\omega_0 n}$; analysis takes it apart and carries the conjugate. A plus sign in both places breaks the identity derived below.

Take the analysis sum over the period that holds $-N_1 \le n \le N_1$. On that period $\tilde{x}[n] = x[n]$. Because x is zero outside $-N_1 \le n \le N_1$, extending the sum to all n adds only zero terms:

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} x[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-jk\omega_0 n}.$$

The last sum has the same form for every k. Define it as a function of the continuous variable ω, and then evaluate it at $\omega = k\omega_0$.

THE ENVELOPE

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \qquad \Longrightarrow \qquad N\, a_k = X(e^{jk\omega_0})$$

$X(e^{j\omega})$ is defined for every real ω and does not depend on N. The scaled coefficients Na_k are its samples, spaced $\omega_0 = 2\pi/N$ apart; a larger N only makes the spacing smaller. The identity holds only because the two exponents match, which is the check on the sign. For the pulse with $N_1 = 2$ and $N = 10$, $10\, a_0 = X(e^{j0}) = \sum_n x[n] = 2N_1 + 1 = 5$, and the same 5 comes out at every period N.

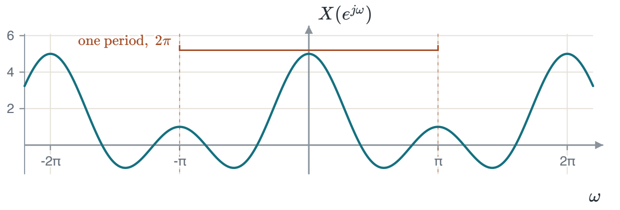

FIGURE 6.4 The envelope of the pulse with $N_1 = 2$, over two periods. Its value at $\omega = 0$ is $2N_1 + 1 = 5$.

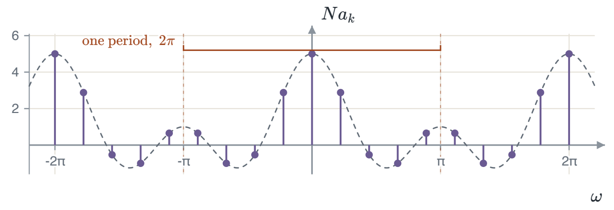

FIGURE 6.5 The scaled coefficients Na_k for $N = 9$: samples of the dashed envelope at $\omega = k\omega_0$.

Letting the period grow

Put $a_k = \frac{1}{N} X(e^{jk\omega_0})$ back into the synthesis equation and write $\frac{1}{N}$ as $\frac{\omega_0}{2\pi}$, which holds because $\omega_0 = 2\pi/N$:

$$\begin{aligned}\tilde{x}[n] &= \sum_{k=\langle N\rangle} \frac{1}{N} X(e^{jk\omega_0}) e^{jk\omega_0 n} \\ &= \frac{1}{2\pi} \sum_{k=\langle N\rangle} X(e^{jk\omega_0}) e^{jk\omega_0 n}\, \omega_0.\end{aligned}$$

This is exact at every N. Read the right-hand side as a sum of rectangles. Each rectangle has height $X(e^{jk\omega_0})e^{jk\omega_0 n}$ and width ω_0. The N rectangles cover an interval of length $N\omega_0 = 2\pi$. As N grows without bound, $\tilde{x}[n]$ approaches $x[n]$. At the same time the width $\omega_0 = 2\pi/N$ approaches zero and becomes $\mathrm{d}\omega$, so the rectangle sum approaches an integral over an interval of length 2π.

THE DISCRETE-TIME FOURIER TRANSFORM PAIR

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \qquad \text{(analysis)}$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega})\, e^{j\omega n}\, \mathrm{d}\omega \qquad \text{(synthesis)}$$

We write the pair $x[n] \leftrightarrow X(e^{j\omega})$. Analysis takes a sequence to its spectrum: the sum over all n leaves a function of ω. Synthesis rebuilds the sequence: the integral over one period leaves a sequence in n. The names go by what each equation does, never by which is written first. Three details separate this pair from the continuous-time one: the factor $\frac{1}{2\pi}$ sits on the synthesis side, the integral runs over one period rather than the whole line, and the left-hand side is written $X(e^{j\omega})$ to record that it is a function of $e^{j\omega}$ and therefore repeats.

⊗ TWO WAYS TO WRITE THE SYNTHESIS EQUATION WRONGLY

The line $x[n] = \int_{-\infty}^{\infty} X(e^{j\omega})e^{j\omega n}\, \mathrm{d}\omega$ has two errors: the factor $\frac{1}{2\pi}$ is missing, and the range must be one period of length 2π. An integral from -2π to 2π covers two periods and returns $2x[n]$, because each period is counted once.

When the transform exists

A sufficient condition is that x is absolutely summable:

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty.$$

Then every term of the analysis sum has modulus $|x[n]e^{-j\omega n}| = |x[n]|$, so the sum converges at every ω and $X(e^{j\omega})$ is continuous. For $x[n] = (0.8)^n u[n]$ the moduli form a geometric series with ratio 0.8: $\sum_{n\ge 0}(0.8)^n = 1/(1-0.8) = 5$. For $(0.5)^n u[n]$ the same formula gives $1/(1-0.5) = 2$. Both transforms exist.

A weaker condition is finite energy, $\sum_n |x[n]|^2 < \infty$. Then the sum converges in mean square: the energy of the error between X and a partial sum goes to zero, and the transform may have jumps. The sequence $x[n] = \sin(\pi n/2)/(\pi n)$, with $x[0] = \frac{1}{2}$, is the example. For odd n, $|\sin(\pi n/2)| = 1$, so $|x[n]| = 1/(\pi|n|)$. The moduli at odd n therefore add like $\frac{2}{\pi}\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots\right)$, and since $\frac{1}{2m+1} \ge \frac{1}{2} \cdot \frac{1}{m+1}$, this is at least half of a harmonic series and has no finite sum. The squares do add up: Example 6.6 shows that x is the inverse transform of a band of height 1 on $|\omega| \le \pi/2$, and Parseval's relation (Section 6.4) gives the energy $\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} 1\, \mathrm{d}\omega = \frac{1}{2}$. That transform jumps at $\omega = \pm\pi/2$.

Finite support is not needed either. It only kept the copies apart in the construction. The pair holds whenever the analysis sum converges.

Why the spectrum repeats

PERIODICITY IN THE FREQUENCY VARIABLE

$$\begin{aligned} X(e^{j(\omega+2\pi)}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega+2\pi)n} \\ &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}e^{-j2\pi n} \\ &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = X(e^{j\omega}) \end{aligned}$$

The whole argument is that n is an integer, so $e^{-j2\pi n} = 1$ in every term of the sum and the extra factor disappears term by term. Any interval of length 2π therefore holds all of X. If $X(e^{j\pi/4}) = 3$, then $X(e^{-j7\pi/4}) = 3$ as well, because $-7\pi/4 = \pi/4 - 2\pi$.

The same step in continuous time gives $\int x(t)e^{-j\omega t}e^{-j2\pi t}\mathrm{d}t$, where t runs over the reals. There $e^{-j2\pi t}$ equals 1 only at integer t, so it cannot leave the integral, and nothing forces $X(j\omega)$ to repeat.

⊗ THE HABIT THIS CHAPTER IS BUILT TO PREVENT

A spectrum drawn on $-\pi \le \omega \le \pi$ alone carries the same information as three periods of it, and it is also the picture a reader later mistakes for a spectrum that stops at $\pm\pi$. Every spectrum here is drawn over more than one period with the period marked.

Frequency in discrete time

The frequency ω of a sequence is measured in radians per sample. A sequence $e^{j\omega n}$ is seen only at integer n, and moving the frequency by 2π changes nothing there:

$$e^{j(\omega+2\pi)n} = e^{j\omega n}\, e^{j2\pi n} = e^{j\omega n}.$$

So ω and $\omega + 2\pi$ give the same sequence. Low frequencies lie near $\omega = 0$ and its copies at multiples of 2π. High frequencies lie near $\omega = \pm\pi$, not at large ω. The fastest sequence is $e^{j\pi n} = (-1)^n$. Of $\cos(0.2\pi n)$, $\cos(1.9\pi n)$ and $\cos(0.9\pi n)$, the last changes fastest from sample to sample: 1.9π is -0.1π after a shift of 2π, so it is a slow sequence, and only 0.9π lies near $\pm\pi$.

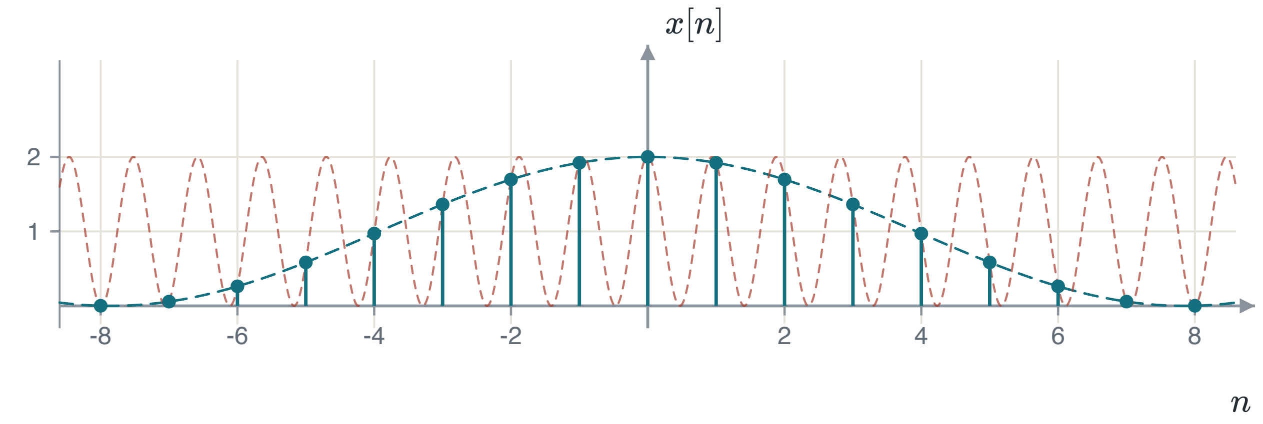

FIGURE 6.6 The stems are $x[n] = 1 + \cos(0.4n)$. The curves $1 + \cos(0.4t)$ and $1 + \cos((0.4 + 2\pi)t)$ both pass through every stem, so the sequence cannot tell the two frequencies apart.

Frequency on the unit circle

The value $e^{j\omega n}$ is a point on the unit circle at angle ωn. One step in n multiplies it by $e^{j\omega}$:

$$e^{j\omega(n+1)} = e^{j\omega n}\, e^{j\omega}.$$

Multiplying by $e^{j\omega}$ turns a point by the angle ω. So the frequency is the turn per sample, and the real part of the point, dropped onto the real axis, is the sample $\cos(\omega n)$. Three facts can be read off the circle.

- A turn of $\omega + 2\pi$ is one full turn more than ω and lands on the same point, so it gives the same samples.
- At $\omega = \pi$ the point jumps between 1 and -1, so the samples are $(-1)^n$. No sequence changes faster.
- A turn of $2\pi - \omega$ is a turn of $-\omega$ plus a full turn. The point turns the other way, $e^{j(2\pi-\omega)n} = e^{-j\omega n}$, which is the complex conjugate of $e^{j\omega n}$. Its real part is the same: $\cos((2\pi - \omega)n) = \cos(2\pi n - \omega n) = \cos(-\omega n) = \cos(\omega n)$, because $2\pi n$ is a whole number of turns and the cosine is even. For example, a turn of $3\pi/2$ per step gives the same samples $\cos(\omega n)$ as a turn of $2\pi - 3\pi/2 = \pi/2$.

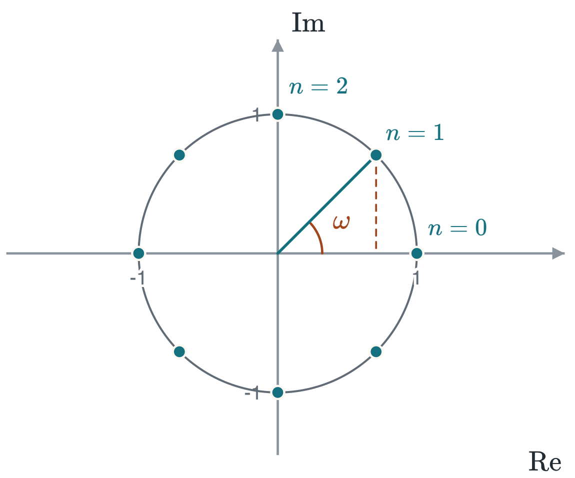

FIGURE 6.7 The points $e^{j\omega n}$ for $\omega = \pi/4$: each step turns by ω, and after eight steps the point is back at 1. The dashed drop gives the real part at $n = 1$.

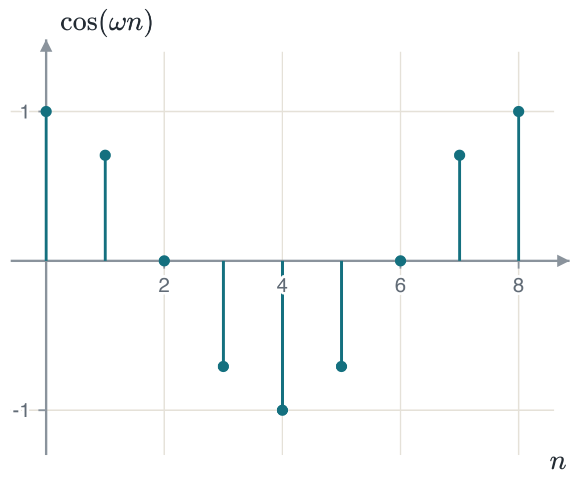

FIGURE 6.8 The real parts, $\cos(\pi n/4)$ for $n = 0, \ldots, 8$. A turn of $7\pi/4$ per step would give the same stems.

The discrete Fourier transform

A computer cannot store a function of a continuous ω. It stores a list of numbers. The discrete Fourier transform, or DFT, is the list of N equally spaced samples of $X(e^{j\omega})$ for a sequence that is zero outside $0 \le n \le N-1$. The first example is the rectangular window of L ones, $w[n] = 1$ for $0 \le n \le L-1$ and 0 elsewhere. Its transform is a finite geometric sum with ratio $r = e^{-j\omega}$, first index $p = 0$ and last index $q = L-1$. The formula $\sum_{n=p}^{q} r^n = (r^p - r^{q+1})/(1-r)$ needs only $r \ne 1$, that is ω not a multiple of 2π. Then balance each difference about its middle exponent:

$$\begin{aligned} W(e^{j\omega}) &= \sum_{n=0}^{L-1} e^{-j\omega n} = \frac{1-e^{-j\omega L}}{1-e^{-j\omega}} \\ &= \frac{e^{-j\omega L/2}\left(e^{j\omega L/2} - e^{-j\omega L/2}\right)}{e^{-j\omega/2}\left(e^{j\omega/2} - e^{-j\omega/2}\right)} \\ &= \frac{e^{-j\omega L/2} \cdot 2j\sin(\omega L/2)}{e^{-j\omega/2} \cdot 2j\sin(\omega/2)} \\ &= e^{-j\omega(L-1)/2}\,\frac{\sin(\omega L/2)}{\sin(\omega/2)}. \end{aligned}$$

The third line uses $e^{j\theta} - e^{-j\theta} = 2j\sin\theta$ twice. At $\omega = 0$ every term of the sum is 1, so $W(e^{j0}) = L$. The magnitude $|\sin(\omega L/2)/\sin(\omega/2)|$ first reaches zero where $\omega L/2 = \pm\pi$, at $\omega = \pm 2\pi/L$. The main lobe between these two zeros is $4\pi/L$ wide.

THE DFT: N SAMPLES OF THE DTFT

$$X[k] = X(e^{j2\pi k/N}) = \sum_{n=0}^{N-1} x[n]\,e^{-j2\pi kn/N}, \qquad k = 0, 1, \dots, N-1$$

The second equality puts $\omega = \omega_k = 2\pi k/N$ into the analysis sum and drops the terms outside $0 \le n \le N-1$, where $x[n] = 0$. Only N values are distinct: $\omega_{k+N} = \omega_k + 2\pi$, so $X[k+N] = X[k]$ by periodicity. The fast Fourier transform, the function fft in MATLAB and in NumPy, computes these N numbers.

The same numbers are the series coefficients of the periodic extension. Repeat $x[n]$ every N samples to get $\tilde{x}$. On $0 \le n \le N-1$ there is only one copy, so $\tilde{x}[n] = x[n]$ there, and the analysis sum of the series over that period gives

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} \tilde{x}[n]\,e^{-jk(2\pi/N)n} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\,e^{-j2\pi kn/N} = \frac{X[k]}{N}.$$

EXAMPLE 6.1 — THE DFT OF FOUR ONES

GIVEN: $x[n] = 1$ for $0 \le n \le 3$ and 0 elsewhere.

FIND: $X(e^{j\omega})$, and the DFT values $X[k]$ for $N = 8$ and for $N = 4$.

METHOD: The sequence is the rectangular window with $L = 4$, so its transform is the formula above. The DFT is that transform sampled at $\omega_k = 2\pi k/N$.

SOLUTION: With $L = 4$:

$$X(e^{j\omega}) = e^{-j3\omega/2}\,\frac{\sin(2\omega)}{\sin(\omega/2)}, \qquad X(e^{j0}) = 4.$$

The factor $e^{-j3\omega/2}$ has modulus 1, so $|X(e^{j\omega})| = |\sin(2\omega)/\sin(\omega/2)|$. For $N = 8$ the samples are at $\omega_k = \pi k/4$, where $2\omega_k = \pi k/2$ and $\omega_k/2 = \pi k/8$:

$$|X[k]| = \left|\frac{\sin(\pi k/2)}{\sin(\pi k/8)}\right|, \qquad k = 1, \ldots, 7.$$

This gives $|X[1]| = 1/\sin(\pi/8) = 1/0.38268 = 2.6131$, $X[2] = 0$ because $\sin\pi = 0$, $|X[3]| = 1/\sin(3\pi/8) = 1/0.92388 = 1.0824$ and $X[4] = 0$; the values for $k = 5, 6, 7$ repeat these in reverse order, since $|\sin(\pi(8-k)/2)| = |\sin(\pi k/2)|$ and $\sin(\pi(8-k)/8) = \sin(\pi k/8)$. With $X[0] = 4$ the list is $4,\ 2.6131,\ 0,\ 1.0824,\ 0,\ 1.0824,\ 0,\ 2.6131$. For $N = 4$ the samples are at $\omega_k = \pi k/2$, where $\sin(2\omega_k) = \sin(\pi k) = 0$ for $k = 1, 2, 3$. So $X[k] = 4,\ 0,\ 0,\ 0$.

CHECK With $N = 4$ the periodic extension of four ones is the constant 1. Its only nonzero series coefficient is $a_0 = 1$, and indeed $a_k = X[k]/4 = 1, 0, 0, 0$.

READING With $N = 4$ every sample except the first lands on a zero of the curve. The four numbers are correct, but they hide the shape of $X(e^{j\omega})$. In the same way, five ones with $N = 5$ give $X[0] = 5$ and $X[1] = \cdots = X[4] = 0$.

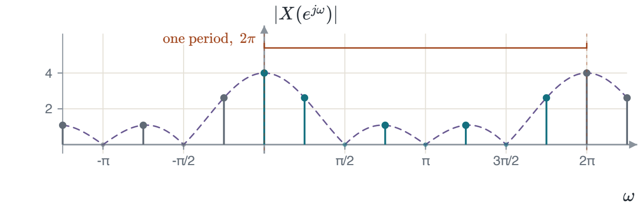

FIGURE 6.9 $N = 8$: the stems are $|X[k]|$ at $\omega_k = 2\pi k/8$, strong for $k = 0, \ldots, 7$ and faint where they repeat. They lie on the dashed curve $|X(e^{j\omega})|$.

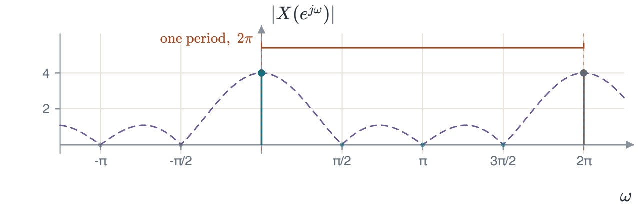

FIGURE 6.10 $N = 4$: the stems read $4, 0, 0, 0$. Three of them land on zeros of the curve.

Zero padding and resolution

Appending zeros to a record of L samples raises N but adds only zero terms to the analysis sum. So $X(e^{j\omega})$ does not change. Zero padding gives more samples, closer together, of the same curve. It adds points, not detail.

Detail comes from the record length L. A cosine kept for L samples has a transform made of copies of $W(e^{j\omega})$ centred on its frequencies, as Section 6.5 shows. Each copy has a main lobe $\Delta\omega = 4\pi/L$ wide, and only more signal makes it narrower. Take the record $x[n] = \cos(0.3\pi n) + \cos(0.4\pi n)$, $0 \le n \le L-1$, whose two frequencies are 0.1π apart. At $L = 12$ each lobe is $4\pi/12 = \pi/3$ wide, more than three times the spacing, and the two lobes merge into one peak. At $L = 40$ each lobe is $4\pi/40 = 0.1\pi$ wide and the two peaks stand apart. No amount of padding separates the peaks of the 12-sample record, because padding does not change its curve.

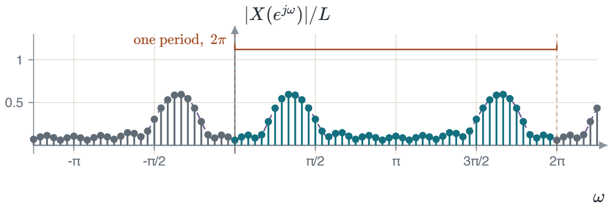

FIGURE 6.11 $L = 12$, padded to $N = 48$: many stems, one merged peak near 0.35π and its mirror.

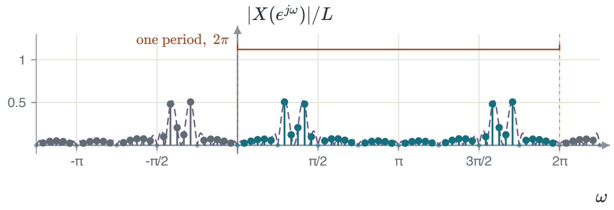

FIGURE 6.12 $L = 40$, padded to $N = 48$: two separate peaks, at 0.3π and 0.4π.

6.2 The standard pairs

Most problems start from one of a few standard sequences: a single sample, a one-sided or two-sided decay, a window, and the inverse of an ideal band. This section derives their transforms. A property from Section 6.4 then turns a standard pair into the transform that a problem needs.

EXAMPLE 6.2 — A SHIFTED UNIT SAMPLE

GIVEN $x[n] = \delta[n - n_0]$, a unit sample at the integer index n_0.

FIND $X(e^{j\omega})$, its magnitude and its phase.

METHOD Use the analysis sum because the sequence is given in time. The unit sample makes every term with $n \neq n_0$ equal to zero, so only one term remains.

SOLUTION Start from the analysis sum and substitute the sequence. The unit sample $\delta[n - n_0]$ is 1 at $n = n_0$ and 0 at every other n, so the sum keeps one term:

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} \delta[n - n_0] e^{-j\omega n} \\ &= e^{-j\omega n_0}. \end{aligned}$$

Write the result in polar form. A complex exponential $e^{j\phi}$ with real ϕ has modulus 1 and angle ϕ, so

$$|X(e^{j\omega})| = |e^{-j\omega n_0}| = 1, \qquad \angle X(e^{j\omega}) = -n_0\omega.$$

The magnitude is 1 at every frequency and the phase is a straight line of slope $-n_0$.

CHECK At $n_0 = 0$ the sequence is $\delta[n]$ and the transform is the constant 1, which is what the sum gives directly.

READING A straight line is not periodic, and yet the transform is. A phase plot shows the principal value of the angle, in $(-\pi, \pi]$. The line $-n_0\omega$ drops by 2π every $2\pi/|n_0|$ in ω, and each drop is folded back, so the plot is a sawtooth of period $2\pi/|n_0|$.

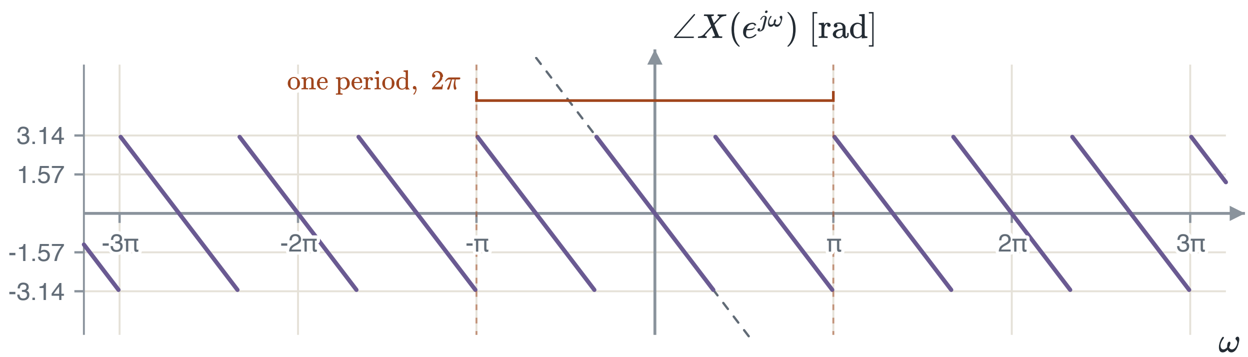

FIGURE 6.13 The phase of $e^{-j3\omega}$, the transform of $\delta[n-3]$. The dashed line is the unwrapped -3ω; the sawtooth is its principal value, of period $2\pi/3$. Both describe the same transform.

EXAMPLE 6.3 — THE ONE-SIDED EXPONENTIAL

GIVEN $x[n] = a^n u[n]$ with a real and $|a| < 1$.

FIND $X(e^{j\omega})$, and the extremes of its magnitude and phase.

METHOD Use the geometric-series formula because the one-sided sequence produces powers of the same ratio $ae^{-j\omega}$. The stated condition makes the infinite sum converge.

SOLUTION Start from the analysis sum and substitute the sequence. The factor $u[n]$ is 0 for $n < 0$ and 1 for $n \geq 0$, so the lower limit moves to $n = 0$. Then collect the two powers of n into one power:

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} a^n u[n]\, e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} a^n e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} \left(ae^{-j\omega}\right)^n. \end{aligned}$$

This is an infinite geometric series with first term 1 and ratio $r = ae^{-j\omega}$. The formula $\sum_{n=0}^{\infty} r^n = 1/(1-r)$ holds when $|r| < 1$. Here $|r| = |a|\,|e^{-j\omega}| = |a| \cdot 1 = |a| < 1$, so the series converges at every ω and

$$a^n u[n] \;\longleftrightarrow\; \frac{1}{1 - ae^{-j\omega}}, \qquad |a| < 1.$$

For $|a| \geq 1$ the terms do not shrink and there is no transform. For the magnitude, write the denominator in rectangular form with $e^{-j\omega} = \cos\omega - j\sin\omega$:

$$1 - ae^{-j\omega} = (1 - a\cos\omega) + j\,a\sin\omega.$$

Its squared modulus is the sum of the squared real and imaginary parts:

$$\begin{aligned}|1-ae^{-j\omega}|^2 &= (1-a\cos\omega)^2 + a^2\sin^2\omega \\ &= 1-2a\cos\omega + a^2\cos^2\omega + a^2\sin^2\omega \\ &= 1-2a\cos\omega + a^2,\end{aligned}$$

using $\cos^2\omega + \sin^2\omega = 1$. The modulus of a quotient is the quotient of the moduli, and the angle of $1/D$ is minus the angle of D. For $|a| < 1$ the real part $1 - a\cos\omega$ is positive, so the arctangent gives the angle of D directly:

$$|X(e^{j\omega})| = \frac{1}{\sqrt{1-2a\cos\omega+a^2}},$$
$$\angle X(e^{j\omega}) = -\angle\big(1-ae^{-j\omega}\big) = -\arctan\frac{a\sin\omega}{1-a\cos\omega}.$$

EXTREMES The magnitude is largest where $1-2a\cos\omega+a^2$ is smallest. For $a > 0$ that is at $\cos\omega = 1$, so $\omega = 0$, where the root is $|1-a|$ and $|X|_{\max} = 1/(1-a)$. It is smallest at $\cos\omega = -1$, so $\omega = \pm\pi$, where the root is $|1+a|$ and $|X|_{\min} = 1/(1+a)$. For negative a the two ends swap, so in general $|X|$ runs from $1/(1+|a|)$ to $1/(1-|a|)$. For the phase, find where the ratio $r(\omega) = a\sin\omega/(1-a\cos\omega)$ is stationary. The quotient rule gives

$$r'(\omega) = \frac{a\cos\omega(1-a\cos\omega) - a\sin\omega\cdot a\sin\omega}{(1-a\cos\omega)^2} = \frac{a(\cos\omega - a)}{(1-a\cos\omega)^2},$$

again using $\cos^2\omega + \sin^2\omega = 1$ in the numerator. So the extreme is at $\cos\omega = a$, where $\sin\omega = \pm\sqrt{1-a^2}$ and

$$|r| = \frac{|a|\sqrt{1-a^2}}{1-a^2} = \frac{|a|}{\sqrt{1-a^2}}.$$

A right triangle with opposite side $|a|$ and adjacent side $\sqrt{1-a^2}$ has hypotenuse 1, so $\arctan|r| = \arcsin|a|$ and $\max|\angle X| = \arcsin|a|$. Geometrically, the denominator $1-ae^{-j\omega}$ traces a circle of radius $|a|$ about the point 1, and the ray from the origin that touches this circle makes the largest angle. That ray and the radius to the touching point form the same right triangle.

NUMBERS $a = \frac{1}{2}$: magnitude between $\frac{2}{3}$ and 2, and $\max|\angle X| = \arcsin\frac{1}{2} = \pi/6 = 0.5236$ rad, at $\omega = \pm\pi/3$ where $\cos\omega = \frac{1}{2}$. $a = \frac{1}{8}$: magnitude between $\frac{8}{9} = 0.8889$ and $\frac{8}{7} = 1.1429$, and $\max|\angle X| = \arcsin\frac{1}{8} = 0.1253$ rad.

CHECK At $\omega = 0$ the transform is $\sum_n x[n] = \sum_{n\geq 0} a^n = 1/(1-a)$, the geometric series summed directly. It matches $|X|_{\max}$ for $a > 0$.

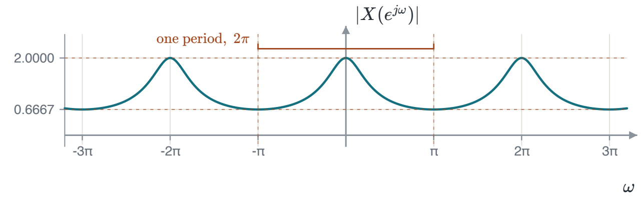

FIGURE 6.14 Magnitude for $a = \frac{1}{2}$, touching $1/(1-a) = 2$ and $1/(1+a) = 2/3$ in every period.

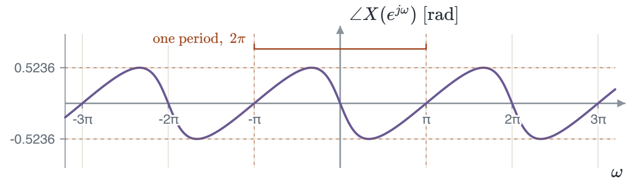

FIGURE 6.15 Phase for $a = \frac{1}{2}$, touching $\pm \arcsin \frac{1}{2} = \pm 0.5236$ rad at $\omega = \mp\pi/3$.

EXAMPLE 6.4 – THE TWO-SIDED EXPONENTIAL

GIVEN $x[n] = a^{|n|}$ with $|a| < 1$.

FIND $X(e^{j\omega})$ and its extremes.

METHOD Split the analysis sum at $n = 0$ because the absolute value changes formula there. Each remaining sum is geometric and converges under the stated condition.

SOLUTION Start from the analysis sum and split it at $n = 0$. For $n \geq 0$, $a^{|n|} = a^n$; for $n \leq -1$, $|n| = -n$ and $a^{|n|} = a^{-n}$:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^{|n|} e^{-j\omega n} = \sum_{n=0}^{\infty} a^n e^{-j\omega n} + \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n}.$$

The first sum is the one-sided pair of Example 6.3, equal to $1/(1 - ae^{-j\omega})$ because $|ae^{-j\omega}| = |a| < 1$. In the second sum put $m = -n$. As n runs from $-\infty$ to -1, m runs from 1 to ∞:

$$\sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n} = \sum_{m=1}^{\infty} a^m e^{j\omega m} = \sum_{m=1}^{\infty} (ae^{j\omega})^m.$$

This is a geometric series with first term $ae^{j\omega}$ and ratio $ae^{j\omega}$, of modulus $|a| < 1$, so it equals $ae^{j\omega}/(1 - ae^{j\omega})$. Add the two parts over the common denominator $(1 - ae^{-j\omega})(1 - ae^{j\omega})$:

$$\begin{aligned} X(e^{j\omega}) &= \frac{1}{1 - ae^{-j\omega}} + \frac{ae^{j\omega}}{1 - ae^{j\omega}} \\ &= \frac{(1 - ae^{j\omega}) + ae^{j\omega}(1 - ae^{-j\omega})}{(1 - ae^{-j\omega})(1 - ae^{j\omega})} \\ &= \frac{1 - ae^{j\omega} + ae^{j\omega} - a^2}{1 - a(e^{j\omega} + e^{-j\omega}) + a^2} \\ &= \frac{1 - a^2}{1 - 2a\cos\omega + a^2}. \end{aligned}$$

In the numerator the two terms $\mp ae^{j\omega}$ cancel and $ae^{j\omega} \cdot ae^{-j\omega} = a^2$. In the denominator, $e^{j\omega} + e^{-j\omega} = 2\cos\omega$.

EXTREMES For $0 < a < 1$, at $\omega = 0$ the value is $\frac{1-a^2}{(1-a)^2} = \frac{(1-a)(1+a)}{(1-a)^2} = \frac{1+a}{1-a}$; at $\omega = \pm\pi$ it is $\frac{1-a^2}{(1+a)^2} = \frac{1-a}{1+a}$. For $a = \frac{1}{2}$ these are 3 and $\frac{1}{3}$.

CHECK — At $\omega = 0$ the transform must equal $\sum_n a^{|n|} = 1 + 2\sum_{n\geq 1} a^n = 1 + \frac{2a}{1-a} = \frac{1+a}{1-a}$. That is 3 for $a = \frac{1}{2}$.

READING — The denominator is $|1 - ae^{-j\omega}|^2$, which is positive, and the numerator $1 - a^2$ is positive for $|a| < 1$. So this spectrum is real and strictly positive, and here $|X| = X$ with $\angle X = 0$ throughout. That is a property of this example, not of real spectra in general.

EXAMPLE 6.5 — THE RECTANGULAR PULSE

GIVEN — $x[n] = 1$ for $-N_1 \leq n \leq N_1$ and zero elsewhere.

FIND — $X(e^{j\omega})$ in closed form, its value at $\omega = 0$ and its zeros.

METHOD — Use the finite geometric-series formula because the pulse has consecutive nonzero samples. Then balance the endpoint exponents so that each difference becomes a sine.

SOLUTION — Start from the analysis sum. The sequence is 1 on $-N_1 \leq n \leq N_1$ and 0 elsewhere, so the limits shrink to the support:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-N_1}^{N_1} e^{-j\omega n} = \sum_{n=-N_1}^{N_1} \left(e^{-j\omega}\right)^n.$$

This is a finite geometric series with ratio $r = e^{-j\omega}$ and $2N_1 + 1$ terms. The finite formula $\sum_{n=p}^{q} r^n = (r^p - r^{q+1})/(1 - r)$ needs only $r \neq 1$. With $p = -N_1$ and $q = N_1$:

$$X(e^{j\omega}) = \frac{e^{j\omega N_1} - e^{-j\omega(N_1+1)}}{1 - e^{-j\omega}}.$$

Now balance the exponents. Multiply numerator and denominator by $e^{j\omega/2}$, which changes nothing because it is the same factor above and below:

$$\begin{aligned} X(e^{j\omega}) &= \frac{e^{j\omega/2}\left(e^{j\omega N_1} - e^{-j\omega(N_1+1)}\right)}{e^{j\omega/2}\left(1 - e^{-j\omega}\right)} \\ &= \frac{e^{j\omega(N_1+\frac{1}{2})} - e^{-j\omega(N_1+\frac{1}{2})}}{e^{j\omega/2} - e^{-j\omega/2}} \\ &= \frac{2j\sin\left(\omega(N_1+\frac{1}{2})\right)}{2j\sin(\omega/2)} \\ &= \frac{\sin\left(\omega(N_1+\frac{1}{2})\right)}{\sin(\omega/2)}. \end{aligned}$$

The third line uses $e^{j\theta} - e^{-j\theta} = 2j\sin\theta$ twice, once with $\theta = \omega(N_1 + \frac{1}{2})$ and once with $\theta = \omega/2$. The factors $2j$ cancel, so the transform is real. This ratio is the **Dirichlet kernel**. With $N = 2N_1 + 1$ samples in the pulse, $N_1 + \frac{1}{2} = N/2$, so the same result reads $\sin(\omega N/2)/\sin(\omega/2)$.

EXCLUDED POINTS — $r = e^{-j\omega} = 1$ at $\omega = 0, \pm 2\pi, \ldots$, which is exactly where the formula becomes $0/0$. Go back to the sum: every term is 1, so $X = 2N_1 + 1$. That is five for $N_1 = 2$ and nine for $N_1 = 4$.

ZEROS — The numerator is zero where $\omega(N_1 + \frac{1}{2}) = \pi k$ for an integer k, that is at

$$\omega = \frac{2\pi k}{2N_1 + 1}.$$

When k is a multiple of $2N_1 + 1$, ω is a multiple of 2π and the value is the peak $2N_1 + 1$ instead. So the zeros are at $2\pi k/(2N_1 + 1)$ with k not a multiple of $2N_1 + 1$. The first zero above $\omega = 0$ is $2\pi/5$ for $N_1 = 2$, $2\pi/7$ for $N_1 = 3$ and $2\pi/9$ for $N_1 = 4$.

CHECK

This is not a sinc. The denominator is $\sin(\omega/2)$, not $\omega/2$, and that is what makes the function periodic: a ratio of two sines repeats every 2π, while the unnormalised sinc, $\operatorname{sinc}\theta = \sin\theta/\theta$, decays and never repeats.

⚠ WHAT A FINITE GEOMETRIC SUM REQUIRES

$\sum_{n=p}^{q} r^n = (r^p - r^{q+1})/(1 - r)$ needs $r \neq 1$ and nothing else. There are finitely many terms, so nothing has to converge; only the division can fail. Here $|r| = 1$ exactly, so a condition $|r| < 1$ would exclude every ω, while $|r| \leq 1$ would admit the one value that breaks it. The condition $|a| < 1$ of Example 6.3 belongs to the infinite sum, where it makes the tail vanish.

Going from $N_1 = 2$ to $N_1 = 4$ the peak rises from 5 to 9, and the first zero moves in from $2\pi/5$ to $2\pi/9$. A wider pulse gives a narrower main lobe, but the lobe does not halve when the pulse length is nearly doubled: the ratio of the first zeros is $\frac{2\pi/9}{2\pi/5} = \frac{5}{9}$, the inverse ratio of the pulse lengths 5 and 9. The period stays 2π.

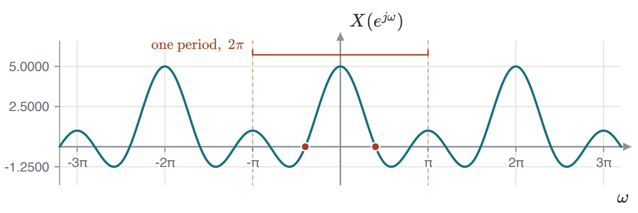

FIGURE 6.16 $N_1 = 2$: peak 5, least value -1.2500, first zeros at $\pm 2\pi/5$ (red).

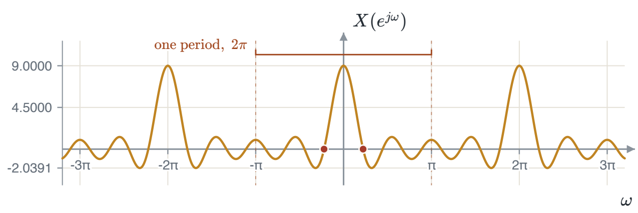

FIGURE 6.17 $N_1 = 4$: peak 9, least value -2.0391, first zeros at $\pm 2\pi/9$.

⊗ REAL IS NOT THE SAME AS NON-NEGATIVE

The kernel above is real at every ω and negative on part of every period. Realness says that the imaginary part is zero. It says nothing about the sign, and the angle depends on the sign:

$$X \text{ real}: \qquad \angle X = \begin{cases} 0, & X > 0 \\ \pi, & X < 0 \end{cases} \qquad |X| = \begin{cases} X, & X \geq 0 \\ -X, & X < 0. \end{cases}$$

For $N_1 = 2$ the kernel at $\omega = 0.6\pi$ is $\sin(1.5\pi)/\sin(0.3\pi) = -1/0.8090 = -1.236$, so there $\angle X = \pi$, not 0. Example 6.4 is the contrasting case: there the spectrum is real **and** strictly positive, so the phase really is zero. Positivity gives zero phase; realness alone does not.

EXAMPLE 6.6 – THE IDEAL LOW-PASS SEQUENCE

GIVEN $X(e^{j\omega}) = 1$ for $|\omega| \le W$ and 0 for $W < |\omega| \le \pi$, repeated with period 2π, with $0 < W \le \pi$.

FIND $x[n]$.

METHOD Use the synthesis integral because the spectrum is given and the sequence is required. Choose the period $-\pi \le \omega \le \pi$ so that the band where the spectrum is 1 appears once in it.

SOLUTION Start from the synthesis integral over the period $-\pi \le \omega \le \pi$. The spectrum is 1 on $|\omega| \le W$ and 0 on the rest of the period, so the limits shrink to $-W \le \omega \le W$:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,\mathrm{d}\omega = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega n}\,\mathrm{d}\omega.$$

For $n \neq 0$ the antiderivative of $e^{j\omega n}$ in ω is $e^{j\omega n}/(jn)$:

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-W}^{W} \\ &= \frac{1}{2\pi}\cdot\frac{e^{jWn} - e^{-jWn}}{jn} \\ &= \frac{1}{2\pi}\cdot\frac{2j\sin(Wn)}{jn} \\ &= \frac{\sin(Wn)}{\pi n}. \end{aligned}$$

The third line uses $e^{j\theta} - e^{-j\theta} = 2j\sin\theta$ with $\theta = Wn$, and then the j and the 2 cancel. At $n = 0$ the antiderivative does not apply because it divides by n. There the integrand is $e^0 = 1$, so the integral is the length $2W$ of the band and

$$x[0] = \frac{1}{2\pi}\cdot 2W = \frac{W}{\pi},$$

which is also the limit of $\sin(Wn)/(\pi n)$ as $n \to 0$.

NUMBERS $W = \pi/4$: 0.25, 0.225079, 0.159155, 0.075026 at $n = 0, 1, 2, 3$. $W = \pi/2$: 0.5, 0.318310, 0, -0.106103; in particular $x[2] = \sin(\pi)/(2\pi) = 0$.

CHECK $x[0]$ must be the fraction W/π of the period that the band occupies. That single number catches both a lost 2π and a wrong prefactor.

READING A narrow band gives a slowly decaying sequence, and a wide band a short one. At $W = \pi$ the band fills the whole period, $x[0] = 1$, and $x[n] = \sin(\pi n)/(\pi n) = 0$ for every $n \neq 0$: the sequence is $\delta[n]$, whose transform is the constant 1.

> ⓘ THE SINC CONVENTION
>
> This course uses the unnormalised sinc, $\operatorname{sinc}\theta = \sin\theta/\theta$, with no π inside the argument. In that convention the same sequence reads $x[n] = \frac{W}{\pi}\operatorname{sinc}(Wn)$. The other common definition puts a π inside the argument and moves every zero crossing, so the convention is restated at every point of use. The prefactor is the constant W/π. Writing $\sin(Wn)/n$ drops the π and is π times too large at every index: it gives 0.707107 at $n = 1$ for $W = \pi/4$, where the true value is 0.225079.

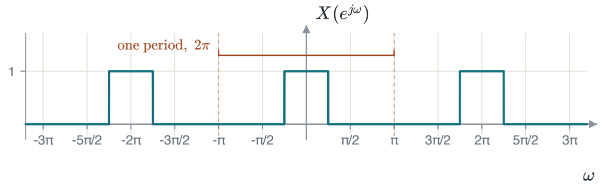

FIGURE 6.18 The spectrum for $W = \pi/4$, over three periods. The band repeats; it does not stop at $\pm\pi$.

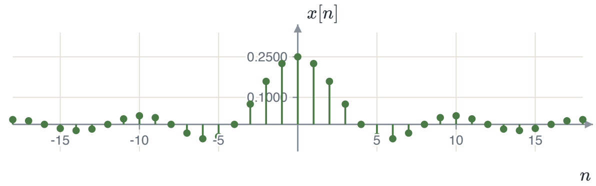

FIGURE 6.19 Its inverse transform, with $x[0] = W/\pi = 0.25$.

6.3 Periodic sequences

A complex exponential is not absolutely summable: $\sum_n |e^{j\omega_0 n}| = \sum_n 1$ diverges. Its analysis sum does not converge in the ordinary sense. Its transform exists as a train of impulses, and the definition is that the synthesis equation returns the sequence.

> THE COMPLEX EXPONENTIAL
>
> $$e^{j\omega_0 n} \;\longleftrightarrow\; X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\,\delta(\omega - \omega_0 - 2\pi k)$$
>
> To confirm the pair, put the impulse train into the synthesis equation. Choose the integration period $\omega_0 - \pi < \omega \le \omega_0 + \pi$, so that it contains exactly one impulse of the train, the one at $\omega = \omega_0$ with $k = 0$. Then the sifting property picks out the integrand at $\omega = \omega_0$:
>
> $$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{\omega_0-\pi}^{\omega_0+\pi} \sum_k 2\pi\,\delta(\omega - \omega_0 - 2\pi k)\,e^{j\omega n}\,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{\omega_0-\pi}^{\omega_0+\pi} 2\pi\,\delta(\omega - \omega_0)\,e^{j\omega n}\,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\cdot 2\pi\,e^{j\omega_0 n} = e^{j\omega_0 n}. \end{aligned}$$
>
> The copies must be there: a single impulse is not 2π-periodic, and a sequence cannot tell ω_0 from $\omega_0 + 2\pi$. With $\omega_0 = 0$ the pair gives the constant: $1 \leftrightarrow 2\pi\sum_k \delta(\omega - 2\pi k)$. A factor in front scales every weight: $3e^{j(\pi/2)n}$ has impulses of weight $3\cdot 2\pi = 6\pi$.

ⓘ WEIGHT, NOT HEIGHT

An impulse has no value at a point; it has a weight, the number that comes out when it is integrated. Every figure here draws an impulse as an arrow whose height **is** its weight, so the two can be read off the same axis. Arrows drawn to a fixed height with the weight written beside them hide exactly the comparison these pictures are for.

The transform of a periodic sequence

A sequence of period N has a series $x[n] = \sum_{k=0}^{N-1} a_k e^{jk(2\pi/N)n}$. By linearity, transform it term by term, using the complex-exponential pair with $\omega_0 = 2\pi k/N$ for the k-th term. Write l for the index of the copies so that it is not confused with k:

$$\begin{aligned} X(e^{j\omega}) &= \sum_{k=0}^{N-1} a_k \sum_{l=-\infty}^{\infty} 2\pi\,\delta\left(\omega - \frac{2\pi k}{N} - 2\pi l\right) \\ &= \sum_{k=0}^{N-1} \sum_{l=-\infty}^{\infty} 2\pi a_k\,\delta\left(\omega - \frac{2\pi(k+lN)}{N}\right). \end{aligned}$$

Now put $m = k + lN$. As k runs over $0, \ldots, N-1$ and l over all integers, m runs over every integer exactly once. The coefficients repeat with period N, so $a_k = a_{k+lN} = a_m$. The double sum becomes one sum over m, which we rename k:

A PERIODIC SEQUENCE

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}n} \longleftrightarrow X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k\,\delta\left(\omega - \frac{2\pi k}{N}\right)$$

The impulses are $2\pi/N$ apart, so one period holds exactly N of them, and their weights are $2\pi a_k$. A sequence of period $N = 6$ has impulses $2\pi/6 = \pi/3$ apart.

✎ EXAMPLE 6.7 — A SEQUENCE OF PERIOD FOUR

GIVEN $x[n] = 1 + \cos(\pi n/2) + \frac{1}{4}(-1)^n$.

FIND The period, the series coefficients and $X(e^{j\omega})$.

METHOD Write every term as an exponential $e^{jk(2\pi/N)n}$ with $0 \le k \le N-1$, read off a_k, and give each impulse the weight $2\pi a_k$.

SOLUTION The terms have periods 1, 4 and 2, so $N = 4$ and $2\pi/N = \pi/2$. The constant is $1 = e^{j0n}$, the term $k = 0$. By Euler's relation $\cos(\pi n/2) = \frac{1}{2}e^{j\pi n/2} + \frac{1}{2}e^{-j\pi n/2}$. The first exponential is the term $k = 1$. The second is moved into the range $0 \le k \le 3$ by a whole number of turns: $e^{-j\pi n/2} = e^{-j\pi n/2}e^{j2\pi n} = e^{j3\pi n/2}$, the term $k = 3$. Finally $(-1)^n = e^{j\pi n}$, the term $k = 2$. So

$$a_0 = 1, \qquad a_1 = \tfrac{1}{2}, \qquad a_2 = \tfrac{1}{4}, \qquad a_3 = \tfrac{1}{2}.$$

The impulses sit at $\omega = k\pi/2$ with weights $2\pi a_k$: 2π at 0, π at $\pi/2$, $\pi/2$ at π, π at $3\pi/2$, and the pattern repeats every 2π.

CHECK

The synthesis sum at $n=0$ is $\sum_k a_k = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{2} = 2.25$, and the sequence gives $x[0] = 1+1+\frac{1}{4} = 2.25$.

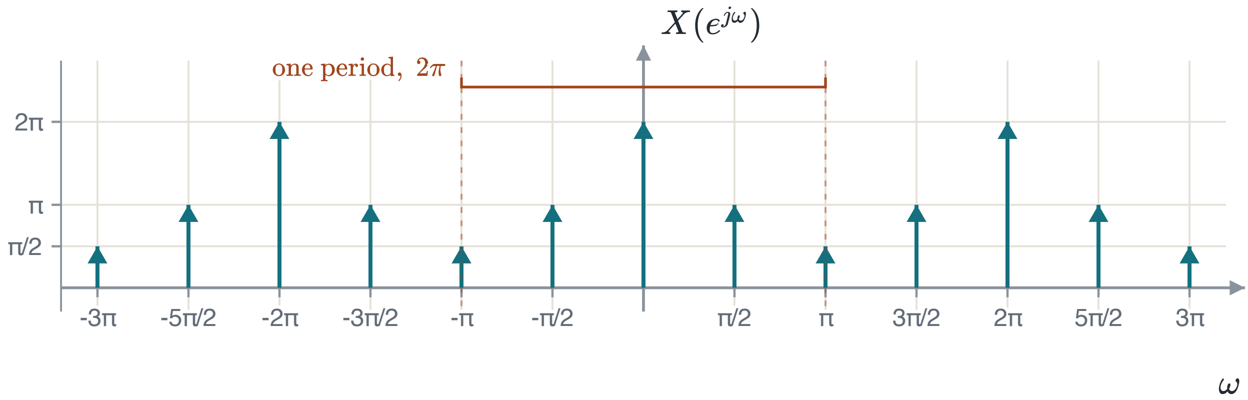

FIGURE 6.20 The transform of $1+\cos(\pi n/2)+\frac{1}{4}(-1)^n$: four impulses in each period, $\pi/2$ apart, of weights 2π, π, $\pi/2$, π.

Cosine and sine

Euler's relations write each as two exponentials, $\cos\omega_0 n = \frac{1}{2}e^{j\omega_0 n} + \frac{1}{2}e^{-j\omega_0 n}$ and $\sin\omega_0 n = \frac{1}{2j}e^{j\omega_0 n} - \frac{1}{2j}e^{-j\omega_0 n}$. Each exponential brings an impulse train of weight 2π, and $\frac{1}{2}\cdot 2\pi = \pi$ while $\frac{1}{2j}\cdot 2\pi = \pi/j$:

COSINE AND SINE

$$\cos\omega_0 n \;\longleftrightarrow\; \pi\sum_{k=-\infty}^{\infty}\big[\delta(\omega-\omega_0-2\pi k)+\delta(\omega+\omega_0-2\pi k)\big]$$

$$\sin\omega_0 n \;\longleftrightarrow\; \frac{\pi}{j}\sum_{k=-\infty}^{\infty}\big[\delta(\omega-\omega_0-2\pi k)-\delta(\omega+\omega_0-2\pi k)\big]$$

The sine weight $\pi/j = -j\pi$ is imaginary: $-j\pi$ at $+\omega_0$ and $+j\pi$ at $-\omega_0$. A real sequence always produces the pair at $+\omega_0$ and $-\omega_0$; dropping the negative one rebuilds $\frac{1}{2}e^{j\omega_0 n}$, which is complex, instead of the cosine. An amplitude multiplies the weights: $3\cos(\pi n/3)$ has weight 3π at $\omega = \pm\pi/3$.

The periodic square wave

Take the square wave that is 1 on $|n|\le N_1$ inside each period of length N. Take the analysis sum of the series over the period centred on the origin. The sequence is 1 on $-N_1\le n\le N_1$ and 0 on the rest of that period, so the limits shrink to the pulse:

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\frac{2\pi}{N}n} = \frac{1}{N}\sum_{n=-N_1}^{N_1} e^{-j\left(\frac{2\pi k}{N}\right)n}.$$

The remaining sum is the rectangular-pulse sum of Example 6.5 with ω replaced by $2\pi k/N$. So the coefficients are the Dirichlet kernel sampled at $\omega = 2\pi k/N$ and divided by N:

$$a_k = \frac{1}{N}\frac{\sin\big(2\pi k(N_1+\frac{1}{2})/N\big)}{\sin(\pi k/N)}, \qquad a_k = \frac{2N_1+1}{N}\ \text{when } N \text{ divides } k.$$

The second form covers the excluded points of Example 6.5. When k is a multiple of N, $\sin(\pi k/N) = 0$ and the closed form fails, but every term of the sum is 1, so the sum is $2N_1 + 1$. The impulse weights are $2\pi a_k$.

EXAMPLE 6.8 – THE SQUARE WAVE WITH $N = 10$, $N_1 = 2$

GIVEN The square wave that is 1 on $|n| \le 2$ in each period of $N = 10$.

FIND The impulse weights $2\pi a_k$ for $k = 0, \ldots, 5$.

METHOD Put $N = 10$ and $N_1 = 2$ into the coefficient formula. Here $2\pi k(N_1 + \frac{1}{2})/N = \pi k/2$ and $\pi k/N = \pi k/10$.

SOLUTION

$$\begin{aligned}
2\pi a_0 &= 2\pi \cdot \tfrac{5}{10} = \pi = 3.1416,\\
2\pi a_1 &= \frac{2\pi}{10} \cdot \frac{\sin(\pi/2)}{\sin(\pi/10)} = \frac{2\pi}{10} \cdot \frac{1}{0.30902} = 2.0333,\\
2\pi a_2 &= \frac{2\pi}{10} \cdot \frac{\sin \pi}{\sin(\pi/5)} = 0,\\
2\pi a_3 &= \frac{2\pi}{10} \cdot \frac{\sin(3\pi/2)}{\sin(3\pi/10)} = \frac{2\pi}{10} \cdot \frac{-1}{0.80902} = -0.7766,\\
2\pi a_4 &= \frac{2\pi}{10} \cdot \frac{\sin 2\pi}{\sin(2\pi/5)} = 0,\\
2\pi a_5 &= \frac{2\pi}{10} \cdot \frac{\sin(5\pi/2)}{\sin(\pi/2)} = \frac{2\pi}{10} = 0.6283.
\end{aligned}$$

The sequence is real and even, so $a_{-k} = a_k$ and the weights at negative k repeat these.

CHECK The weights are unequal, two of them are zero, and the ones at $\pm 3\pi/5$ are negative, so those arrows point down. Sketching all impulses the same length hides all three facts.

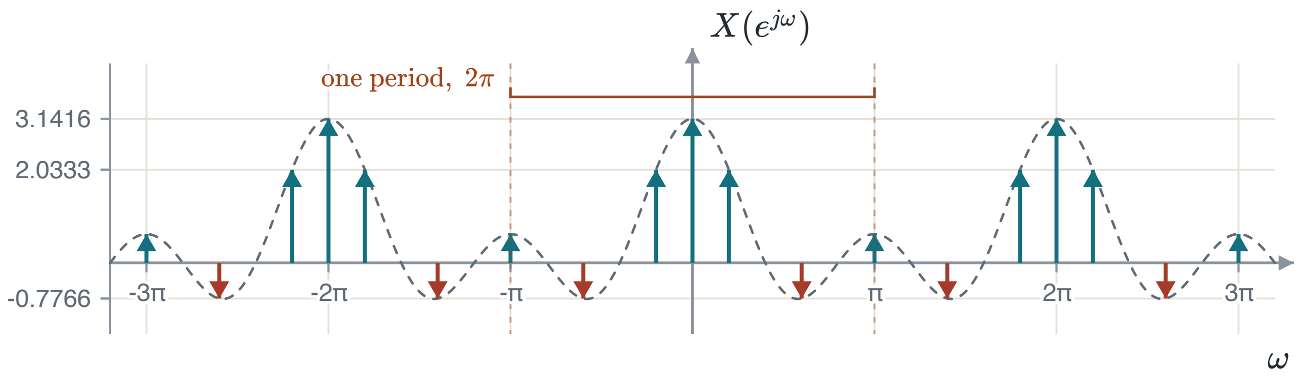

FIGURE 6.21 The square wave with $N = 10$, $N_1 = 2$. The dashed line is the envelope $\frac{2\pi}{N}\sin(\omega(N_1 + \frac{1}{2}))/\sin(\omega/2)$; the red arrows are the negative weights.

The weights lie on one curve. Write $2\pi a_k$ as $\frac{2\pi}{N}$ times the envelope at $\omega = 2\pi k/N$, where the envelope is the Dirichlet kernel, the transform of one pulse. As N grows, the spacing $2\pi/N$ and every weight shrink like $1/N$, while the envelope keeps its shape. With $N_1 = 2$ and $N = 50$ the weight at $\omega = 0$ is $2\pi \cdot 5/50 = \pi/5$. With $N_1 = 3$ and $N = 10$ it is $2\pi \cdot 7/10 = 7\pi/5$.

✎ EXAMPLE 6.9 – THE IMPULSE TRAIN

GIVEN $x[n] = \sum_{m=-\infty}^{\infty} \delta[n - mN]$: a unit sample every N indices. Here n is the sequence index and m labels the copies, so the two roles use different letters.

FIND The coefficients and the transform.

METHOD The sequence is periodic with period N, so its transform is an impulse train. Find a_k over $0 \le n \le N-1$, where only $x[0] = 1$ is nonzero, then give each impulse the weight $2\pi a_k$.

SOLUTION Inside $0 \le n \le N-1$ the only nonzero sample is $x[0] = 1$, so the analysis sum keeps one term:

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\frac{2\pi}{N}n} = \frac{1}{N}\, x[0]\, e^0 = \frac{1}{N} \qquad \text{for every } k.$$

Put $a_k = 1/N$ into the periodic-sequence pair. Every impulse gets the same weight $2\pi a_k = 2\pi/N$:

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi \cdot \frac{1}{N}\, \delta\left(\omega - \frac{2\pi k}{N}\right) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{N}\right).$$

NUMBERS Weights $2\pi/5 = 1.2566$, $2\pi/10 = 0.6283$, $2\pi/15 = 0.4189$ for $N = 5, 10, 15$. If N doubles, each weight halves.

CHECK One period holds N impulses of weight $2\pi/N$, so their weights add to 2π for every N. The synthesis equation at $n = 0$ needs exactly that: $x[0] = \frac{1}{2\pi} \cdot 2\pi = 1$.

READING A sparser train in time gives a denser train in frequency, with smaller impulses. Writing the train as $\sum_n \delta[n - nN]$ is a common error: then n has two jobs in one line, and the copies need their own index.

✎ EXAMPLE 6.10 – TWO COSINES

GIVEN $x[n] = 2\cos\left(\frac{5\pi}{3}n\right) + \cos\left(\frac{7\pi}{4}n\right)$.

FIND The spectrum, and the fundamental period of the sequence.

METHOD First reduce each frequency to $-\pi < \omega \le \pi$, because frequencies 2π apart describe the same sequence. Then transform each cosine and add the results by linearity.

REDUCTION Write $\frac{5\pi}{3} = 2\pi - \frac{\pi}{3}$. The term $2\pi n$ is a whole number of turns at every integer n, and the cosine is even:

$$\cos\left(\frac{5\pi}{3}n\right) = \cos\left(2\pi n - \frac{\pi}{3}n\right) = \cos\left(-\frac{\pi}{3}n\right) = \cos\left(\frac{\pi}{3}n\right).$$

The same two moves with $\frac{7\pi}{4} = 2\pi - \frac{\pi}{4}$ give $\cos\left(\frac{7\pi}{4}n\right) = \cos\left(\frac{\pi}{4}n\right)$. These are not merely similar sequences; they take the same value at every n.

SOLUTION After the reduction, $x[n] = 2\cos(\frac{\pi}{3}n) + \cos(\frac{\pi}{4}n)$. Apply the cosine pair to each term and add by linearity:

$$X(e^{j\omega}) = 2\pi \sum_k \Big[\delta\big(\omega - \tfrac{\pi}{3} - 2\pi k\big) + \delta\big(\omega + \tfrac{\pi}{3} - 2\pi k\big)\Big]$$
$$+ \pi \sum_k \Big[\delta\big(\omega - \tfrac{\pi}{4} - 2\pi k\big) + \delta\big(\omega + \tfrac{\pi}{4} - 2\pi k\big)\Big].$$

The amplitude 2 multiplies the weight π of the first cosine pair to 2π; the second cosine has amplitude 1 and keeps the weight π. Inside one period, $-\pi < \omega \le \pi$, there are four impulses: weight 2π at $\pm\pi/3$ and weight π at $\pm\pi/4$. Every copy 2π away also belongs to the answer: weight 2π at $\pm 5\pi/3$ and $\pm 7\pi/3$, weight π at $\pm 7\pi/4$ and $\pm 9\pi/4$.

PERIOD For a discrete-time frequency ω_0 the fundamental period is $N_0 = (2\pi/\omega_0)m$, where m is the smallest positive integer that makes N_0 an integer; no such m exists when $\omega_0/2\pi$ is irrational. For $\omega_1 = 5\pi/3$: $2\pi/\omega_1 = 6/5$, $m = 5$, $N_0 = 6$. For $\omega_2 = 7\pi/4$: $2\pi/\omega_2 = 8/7$, $m = 7$, $N_0 = 8$. The sum repeats when both terms do, after $\operatorname{lcm}(6, 8) = 24$ samples. With $6 = 2 \cdot 3$ and $8 = 2^3$ the lcm is $2^3 \cdot 3 = 24$; and $24/6 = 4$, $24/8 = 3$ share no factor, so nothing smaller works.

CHECK $x[n]$ is real and even, so $X(e^{j\omega})$ must be real and even. Each impulse at $+\omega$ has a twin of the same weight at $-\omega$. Drawing impulses only at $\pm 5\pi/3$ and $\pm 7\pi/4$ is the common error: inside $-\pi < \omega \le \pi$ they sit at $\pm\pi/3$ and $\pm\pi/4$.

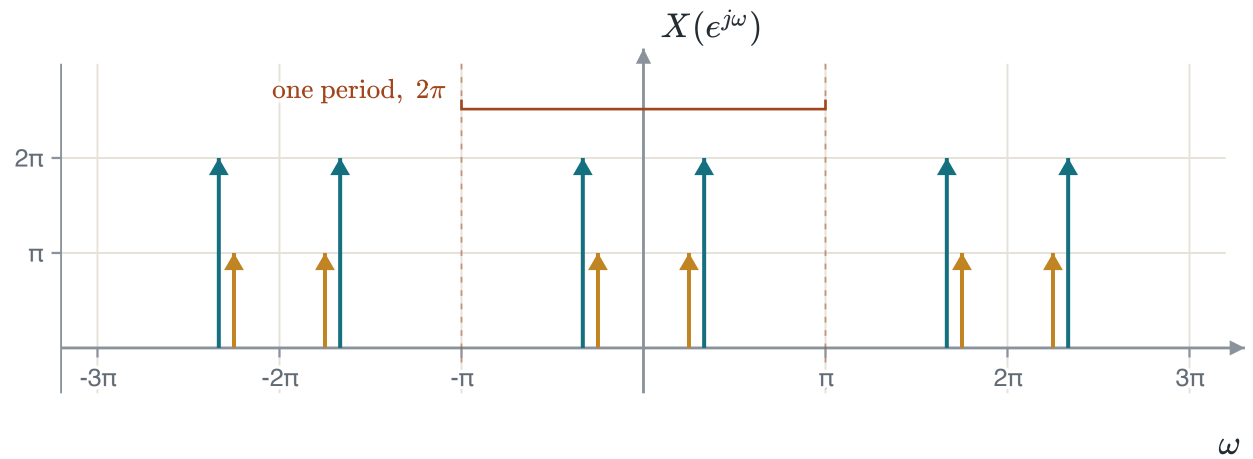

FIGURE 6.22 The spectrum over three periods. Tall arrows carry weight 2π at $\pm\pi/3$, short ones π at $\pm\pi/4$, and each pattern repeats every 2π.

6.4 Properties

Each property is proved the same way: write the analysis sum for the new sequence, substitute, and rearrange until the analysis sum of x appears. Throughout, $x[n] \leftrightarrow X(e^{j\omega})$. The full list is collected in the table of Section 6.7.

Linearity and time shift

The analysis equation is a sum, and a sum is linear:

$$\sum_n \big(a\,x_1[n] + b\,x_2[n]\big)e^{-j\omega n} = a\sum_n x_1[n]e^{-j\omega n} + b\sum_n x_2[n]e^{-j\omega n} = a\,X_1(e^{j\omega}) + b\,X_2(e^{j\omega}).$$

The time shift needs a change of index. Put $m = n - n_0$, so $n = m + n_0$; as n runs over all integers, so does m, and the limits do not change:

$$\begin{aligned}\sum_{n=-\infty}^{\infty} x[n-n_0]e^{-j\omega n} &= \sum_{m=-\infty}^{\infty} x[m]e^{-j\omega(m+n_0)} \\ &= e^{-j\omega n_0}\sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m} = e^{-j\omega n_0}X(e^{j\omega}).\end{aligned}$$

The factor $e^{-j\omega n_0}$ does not depend on m, so it moves outside the sum. It has modulus 1, so a delay leaves $|X|$ unchanged and adds $-\omega n_0$ to the phase. For $y[n] = x[n-2]$ at $\omega = 0.5$, $\angle Y - \angle X = -0.5 \cdot 2 = -1$ rad.

Frequency shift

Multiplying by $e^{j\omega_0 n}$, a sequence of fixed frequency ω_0, needs no change of index; the two exponentials combine into the analysis sum at $\omega - \omega_0$:

$$\sum_n e^{j\omega_0 n}x[n]e^{-j\omega n} = \sum_n x[n]e^{-j(\omega-\omega_0)n} = X\left(e^{j(\omega-\omega_0)}\right).$$

Every copy of the spectrum moves by ω_0. A peak pushed past $\omega = \pi$ comes back in at $-\pi$, because the spectrum repeats every 2π. For example, $(-1)^n = e^{j\pi n}$, so $y[n] = (-1)^n x[n]$ moves a peak at $\omega = 0$ to $\omega = \pi$.

Conjugation and spectral symmetry

Conjugation uses $(ab)^* = a^*b^*$ and $(e^{-j\omega n})^* = e^{j\omega n}$, so conjugating the analysis sum changes the sign of j in every exponent:

$$\sum_n x^*[n]e^{-j\omega n} = \left(\sum_n x[n]e^{j\omega n}\right)^* = \left(\sum_n x[n]e^{-j(-\omega)n}\right)^* = X^*(e^{-j\omega}).$$

For a real sequence, $x^*[n] = x[n]$, so the two transforms are equal and $X(e^{-j\omega}) = X^*(e^{j\omega})$. Taking real parts, moduli, imaginary parts and angles of both sides: the real part and the magnitude are even in ω , and the imaginary part and the phase are odd. So for a real sequence the values on $0 \le \omega \le \pi$ fix those on $-\pi \le \omega < 0$; a complex sequence needs the whole period. As an example, if x is real and $X(e^{j0.4}) = 1.2 - 0.5j$, then periodicity and symmetry give $X\left(e^{j(2\pi-0.4)}\right) = X(e^{-j0.4}) = X^*(e^{j0.4}) = 1.2 + 0.5j$.

Time reversal

Put $m = -n$. As n runs over all integers, so does m, and the result is the analysis sum at $-\omega$:

$$\sum_n x[-n]e^{-j\omega n} = \sum_m x[m]e^{-j\omega(-m)} = \sum_m x[m]e^{-j(-\omega)m} = X(e^{-j\omega}).$$

For a real x, $X(e^{-j\omega}) = X^*(e^{j\omega})$: reversal keeps the magnitude and changes the sign of the phase. If x is even, $x[-n] = x[n]$, reversal changes nothing, and $X(e^{-j\omega}) = X(e^{j\omega})$: the transform is even in ω. Example: $x[n] = \delta[n] + 2\delta[n-1]$ has $X(e^{j\omega}) = 1 + 2e^{-j\omega}$, and $x[-n] = \delta[n] + 2\delta[n+1]$ has $1 + 2e^{j\omega} = X(e^{-j\omega})$.

Even and odd parts

The even part is $\mathcal{E}\mathrm{v}\{x[n]\} = \frac{1}{2}(x[n] + x[-n])$ and the odd part is $\mathcal{O}\mathrm{dd}\{x[n]\} = \frac{1}{2}(x[n] - x[-n])$. By linearity and time reversal, the even part transforms to $\frac{1}{2}[X(e^{j\omega}) + X(e^{-j\omega})]$. For a real sequence $X(e^{-j\omega}) = X^*(e^{j\omega})$, and $\frac{1}{2}(z + z^*) = \mathrm{Re}\{z\}$ for any complex z. The odd part follows the same way with $\frac{1}{2}(z - z^*) = j\,\mathrm{Im}\{z\}$:

EVEN AND ODD PARTS OF A REAL SEQUENCE

$$\mathcal{E}\mathrm{v}\{x[n]\} \longleftrightarrow \tfrac{1}{2}\left[X + X^*\right] = \mathrm{Re}\{X(e^{j\omega})\}, \qquad \mathcal{O}\mathrm{dd}\{x[n]\} \longleftrightarrow \tfrac{1}{2}\left[X - X^*\right] = j\,\mathrm{Im}\{X(e^{j\omega})\}$$

A real and even sequence has no odd part, so its transform is real, and it is even by the symmetry above. A real and odd sequence has no even part, so its transform is purely imaginary and odd. For example, $x[n] = \delta[n+1] - \delta[n-1]$ is real and odd, and $X(e^{j\omega}) = e^{j\omega} - e^{-j\omega} = 2j\sin\omega$, which is $2j$ at $\omega = \pi/2$.

EXAMPLE 6.11 – THE EVEN PART OF A ONE-SIDED SEQUENCE

GIVEN $x[n] = a^n u[n]$ with $0 < a < 1$, so $X(e^{j\omega}) = 1/(1 - ae^{-j\omega})$ by Example 6.3.

FIND The transform of $\mathcal{E}\mathrm{v}\{x[n]\}$, and its values at $\omega = 0$ and $\omega = \pi$ for $a = 0.6$.

METHOD Write the even part in terms of known sequences, transform term by term, and compare with $\mathrm{Re}\{X\}$.

SOLUTION For $n > 0$, $x[-n] = 0$, so the even part is $\frac{1}{2}a^n$. For $n < 0$ it is $\frac{1}{2}a^{-n}$. At $n = 0$ both halves meet: $\frac{1}{2}(x[0] + x[0]) = 1$. In one formula,

$$\mathcal{E}\mathrm{v}\{x[n]\} = \tfrac{1}{2}\left(a^n u[n] + a^{-n}u[-n]\right) = \tfrac{1}{2}a^{|n|} + \tfrac{1}{2}\delta[n],$$

where the δ term supplies the missing $\frac{1}{2}$ at the origin. Transform term by term with Example 6.4 and $\delta[n] \leftrightarrow 1$, then put the two terms over the common denominator:

$$\begin{aligned} \tfrac{1}{2}\cdot\frac{1-a^2}{1-2a\cos\omega+a^2} + \tfrac{1}{2} &= \frac{(1-a^2)+(1-2a\cos\omega+a^2)}{2\,(1-2a\cos\omega+a^2)} \\ &= \frac{2-2a\cos\omega}{2\,(1-2a\cos\omega+a^2)} \\ &= \frac{1-a\cos\omega}{1-2a\cos\omega+a^2}. \end{aligned}$$

For $a = 0.6$: at $\omega = 0$, $\frac{1-0.6}{1-1.2+0.36} = \frac{0.4}{0.16} = 2.5$; at $\omega = \pi$, $\frac{1+0.6}{1+1.2+0.36} = \frac{1.6}{2.56} = 0.625$.

CHECK This is $\mathrm{Re}\{1/(1 - ae^{-j\omega})\}$ exactly. In Example 6.3 the denominator is $(1 - a\cos\omega) + ja\sin\omega$. Multiplying numerator and denominator by its conjugate gives the denominator $1 - 2a\cos\omega + a^2$ and the real part $(1 - a\cos\omega)/(1 - 2a\cos\omega + a^2)$. At $\omega = 0$ both sides are $1/(1-a)$, which is 2.5 for $a = 0.6$.

Time expansion

For an integer $k \geq 1$, the expansion $x_{(k)}[n]$ is $x[n/k]$ when n is a multiple of k and zero otherwise: the same values, with $k-1$ zeros inserted between each pair. Discrete time has no operation that stretches a sequence without leaving gaps. For $x[n] = 1$ on $0 \leq n \leq 2$, the last nonzero sample of $x_{(2)}$ is $x[2]$, moved to $n = 2 \cdot 2 = 4$. In the analysis sum only the indices $n = rk$ with integer r contribute, because every other term is zero. Put $n = rk$; as n runs over the multiples of k, r runs over all integers:

$$\sum_{n=-\infty}^{\infty} x_{(k)}[n]e^{-j\omega n} = \sum_{r=-\infty}^{\infty} x_{(k)}[rk]\, e^{-j\omega rk} = \sum_{r=-\infty}^{\infty} x[r]\, e^{-j(k\omega)r} = X(e^{jk\omega}).$$

The middle step uses $x_{(k)}[rk] = x[rk/k] = x[r]$. The last sum is the analysis sum of x with ω replaced by $k\omega$. That transform repeats every $2\pi/k$:

$$X\left(e^{jk(\omega+2\pi/k)}\right) = X\left(e^{jk\omega+j2\pi}\right) = X\left(e^{jk\omega}\right).$$

So the frequency axis is compressed by k, and one period of length 2π now holds k copies of the old picture. The inserted zeros add no energy, so $x_{(k)}$ has the energy of x, and each copy keeps the height of the original peak. If X has one peak in $-\pi < \omega \le \pi$, at $\omega = 0$, then the transform of $x_{(4)}$ has four there, at $-\pi/2$, 0, $\pi/2$ and π.

EXAMPLE 6.12 – EXPANSION OF A FIVE-POINT PULSE

GIVEN $g[n] = 1$ on $|n| \le 2$, $y[n] = g[n-2]$, and $x[n] = y_{(2)}[n] + 2y_{(2)}[n-1]$.

FIND $X(e^{j\omega})$ and its value at $\omega = 0$.

METHOD Build the sequence in three steps, a shift, an expansion and a weighted sum with a delayed copy, and apply the matching property at each step. Each step changes a known transform, so no new sum is needed.

SOLUTION The pulse is Example 6.5 with $N_1 = 2$, so $G(e^{j\omega}) = \sin(5\omega/2)/\sin(\omega/2)$. The shift by $n_0 = 2$ gives

$$Y(e^{j\omega}) = e^{-j2\omega}G(e^{j\omega}) = e^{-j2\omega}\frac{\sin(5\omega/2)}{\sin(\omega/2)}.$$

Expansion by $k = 2$ replaces ω by 2ω everywhere:

$$Y_{(2)}(e^{j\omega}) = Y(e^{j2\omega}) = e^{-j4\omega}\frac{\sin(5\omega)}{\sin\omega}.$$

Finally, by linearity and the shift by 1,

$$X(e^{j\omega}) = Y_{(2)}(e^{j\omega}) + 2e^{-j\omega}Y_{(2)}(e^{j\omega}) = \left(1 + 2e^{-j\omega}\right)e^{-j4\omega}\frac{\sin(5\omega)}{\sin\omega}.$$

At $\omega = 0$ the first factor is $1 + 2 = 3$, the exponential is 1 and the kernel takes its peak value 5, the number of ones in the pulse. So $X(e^{j0}) = 3 \times 5 = 15$.

CHECK $X(e^{j0}) = \sum_n x[n]$. The expanded sequence has five ones at even indices, and the delayed copy has five twos at odd indices, so $\sum_n x[n] = 5 + 2 \cdot 5 = 15$.

WHERE THE ARGUMENT DOUBLED G carries $\sin(5\omega/2)/\sin(\omega/2)$ and $Y_{(2)}$ carries $\sin(5\omega)/\sin\omega$: the same expression with ω replaced by 2ω. Writing $\sin\omega$ in the denominator of G, $\sin(5\omega/2)/\sin\omega$, is wrong for both sequences. It puts a pole at $\omega = \pi$, where the true value is $G(e^{j\pi}) = \sin(5\pi/2)/\sin(\pi/2) = 1 = \sum_{n=-2}^{2}(-1)^n$; it gives 1000.0 at $\omega = \pi - 10^{-3}$. At $\pi/2$ it gives -0.7071 against the true $\sin(5\pi/4)/\sin(\pi/4) = -1.0000$, and it does not even repeat every 2π.

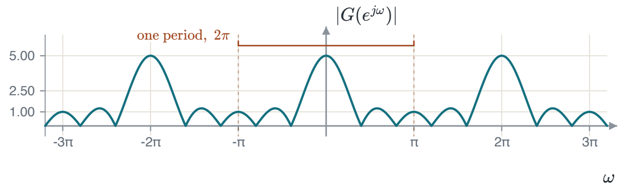

FIGURE 6.23 $|G(e^{j\omega})|$: peak 5, and the finite value 1 at $\omega = \pm\pi$.

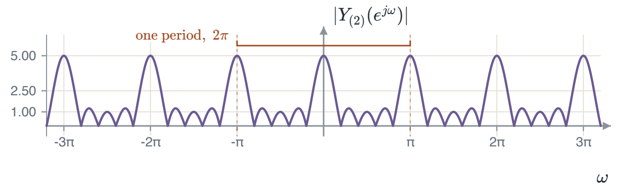

FIGURE 6.24 $|Y_{(2)}(e^{j\omega})|$: the same picture twice inside every 2π, with the same peak 5.

First difference

Discrete time has a **first difference**, $x[n] - x[n-1]$, not a derivative: a sequence has no values between n and $n+1$, so there is no limit to take. Linearity and the time shift with $n_0 = 1$ give its transform, $X(e^{j\omega}) - e^{-j\omega}X(e^{j\omega})$:

$$x[n] - x[n-1] \;\longleftrightarrow\; \left(1 - e^{-j\omega}\right)X(e^{j\omega}).$$

Balance the factor about its middle exponent to read its size: $1 - e^{-j\omega} = e^{-j\omega/2}\left(e^{j\omega/2} - e^{-j\omega/2}\right) = 2j\,e^{-j\omega/2}\sin(\omega/2)$, so $|1 - e^{-j\omega}| = 2|\sin(\omega/2)|$. The factor is 0 at $\omega = 0$ and 2 at $\omega = \pm\pi$. A first difference removes the average and lifts the fastest changes. If $X(e^{j\pi}) = 0.5$, then $Y(e^{j\pi}) = (1 - e^{-j\pi}) \cdot 0.5 = 2 \cdot 0.5 = 1$.

> ⊗ TWO NAMES THAT MUST NOT BE EXCHANGED
>
> Discrete time has a **difference**, not a derivative, and its factor is $1 - e^{-j\omega}$ rather than $j\omega$. The **differentiation** in this chapter is in frequency, and it is a genuine derivative because ω is continuous. Calling the first difference a derivative merges the continuous and discrete cases at the point where they differ.

Accumulation

The running sum $\sum_{m=-\infty}^{n} x[m]$ is the convolution of x with the unit step, because $u[n-m]$ is 1 exactly for $m \le n$: $\sum_m x[m]u[n-m] = \sum_{m \le n} x[m]$. The unit step has the pair

$$u[n] \;\longleftrightarrow\; \frac{1}{1 - e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k).$$

Its two parts have separate reasons. The first difference of $u[n]$ is $\delta[n]$, so by the difference property $(1 - e^{-j\omega})U(e^{j\omega}) = 1$, which fixes $U = 1/(1 - e^{-j\omega})$ wherever $1 - e^{-j\omega} \neq 0$. The difference cannot see a constant, and the step has the average value $\frac{1}{2}$. The constant pair $1 \leftrightarrow 2\pi \sum_k \delta(\omega - 2\pi k)$ gives that average the impulse train of weight $2\pi \cdot \frac{1}{2} = \pi$. By the convolution property (Section 6.5) the transform of the running sum is $X(e^{j\omega})U(e^{j\omega})$. In the impulse part, each impulse at $2\pi k$ keeps only the value $X(e^{j2\pi k}) = X(e^{j0})$, by periodicity:

ACCUMULATION

$$\sum_{m=-\infty}^{n} x[m] \;\longleftrightarrow\; \frac{X(e^{j\omega})}{1-e^{-j\omega}} + \pi X(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

Accumulation undoes differencing, so their factors are reciprocals, and the impulse train is the part that differencing destroys. The running sum settles at $X(e^{j0}) = \sum_n x[n]$, and a constant has an impulse at every multiple of 2π. Where the sequence sums to zero the term vanishes. It is the discrete-time counterpart of the $\pi X(0)\delta(\omega)$ in the integration property of Chapter 5, and it is needed for the same reason: $1 - e^{-j\omega}$ is zero at $\omega = 0$.

Example: for $x[n] = (0.5)^n u[n]$ the running sum at $n \geq 0$ is a finite geometric sum, $\sum_{m=0}^{n} (0.5)^m = \frac{1-(0.5)^{n+1}}{1-0.5} = 2 - (0.5)^n$. It settles at $2 = 1/(1-0.5) = X(e^{j0})$. For $x[n] = (0.75)^n u[n]$, $X(e^{j0}) = 1/(1-0.75) = 4$, so each impulse in the transform of the running sum has weight $\pi X(e^{j0}) = 4\pi$.

Differentiation in frequency

Differentiate the analysis sum term by term in ω. Only $e^{-j\omega n}$ depends on ω, and $\frac{\mathrm{d}}{\mathrm{d}\omega} e^{-j\omega n} = -jn\, e^{-j\omega n}$:

$$\frac{\mathrm{d}X(e^{j\omega})}{\mathrm{d}\omega} = \sum_n x[n](-jn)e^{-j\omega n} = -j \sum_n n\, x[n] e^{-j\omega n} \qquad \Longrightarrow \qquad \sum_n n\, x[n] e^{-j\omega n} = j\, \frac{\mathrm{d}X(e^{j\omega})}{\mathrm{d}\omega}.$$

The last step multiplies both sides by j and uses $j \cdot (-j) = 1$. So $n\, x[n] \leftrightarrow j\, \mathrm{d}X(e^{j\omega})/\mathrm{d}\omega$. Apply it to $a^n u[n]$. The chain rule on $(1 - ae^{-j\omega})^{-1}$ gives

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\left(1 - ae^{-j\omega}\right)^{-1} = -\left(1 - ae^{-j\omega}\right)^{-2} \cdot \left(jae^{-j\omega}\right) = \frac{-jae^{-j\omega}}{\left(1 - ae^{-j\omega}\right)^2},$$

$$n\, a^n u[n] \;\longleftrightarrow\; j \cdot \frac{-jae^{-j\omega}}{\left(1 - ae^{-j\omega}\right)^2} = \frac{ae^{-j\omega}}{\left(1 - ae^{-j\omega}\right)^2}, \qquad |a| < 1.$$

Check at $\omega = 0$ with $a = 0.5$: the pair gives $0.5/(1-0.5)^2 = 0.5/0.25 = 2$, and the sum of the samples is $\sum_{n \geq 0} n(0.5)^n = 2$ as well.

Parseval's relation

Start from the energy sum, write $|x[n]|^2 = x[n]x^*[n]$, and replace $x^*[n]$ by the conjugate of its synthesis integral. Conjugating the integral conjugates X and turns $e^{j\omega n}$ into $e^{-j\omega n}$. Then swap the sum and the integral; the bracket that remains is the analysis sum, which is $X(e^{j\omega})$:

$$\begin{aligned}
\sum_n |x[n]|^2 = \sum_n x[n]\, x^*[n] &= \sum_n x[n]\, \frac{1}{2\pi} \int_{2\pi} X^*(e^{j\omega}) e^{-j\omega n}\, \mathrm{d}\omega \\
&= \frac{1}{2\pi} \int_{2\pi} X^*(e^{j\omega}) \left[\sum_n x[n] e^{-j\omega n}\right] \mathrm{d}\omega \\
&= \frac{1}{2\pi} \int_{2\pi} X^*(e^{j\omega}) X(e^{j\omega})\, \mathrm{d}\omega = \frac{1}{2\pi} \int_{2\pi} |X(e^{j\omega})|^2\, \mathrm{d}\omega.
\end{aligned}$$

Energy is normalised to $R = 1\ \Omega$, as in Chapter 1. The quantity $|X(e^{j\omega})|^2$ is the energy-density spectrum. Integrating it over one period and dividing by 2π gives the total energy; integrating over part of a period gives the energy in that band. Both the one-period range and the factor $\frac{1}{2\pi}$ are needed: over all ω the integral diverges, and without $\frac{1}{2\pi}$ it gives 2π times the energy. For $x[n] = \delta[n] + \delta[n-1]$ the integral must equal $1^2 + 1^2 = 2$.

Two checks in both domains. For $x[n] = a^n u[n]$ with $a = \frac{1}{2}$, the time side is a geometric series with first term 1 and ratio $a^2 = \frac{1}{4}$: $\sum_{n\geq 0}(\frac{1}{4})^n = 1/(1-\frac{1}{4}) = \frac{4}{3}$. On the frequency side, $|X|^2 = 1/(1 - 2a\cos\omega + a^2)$ from Example 6.3. Example 6.4 shows that $(1-a^2)/(1-2a\cos\omega + a^2)$ is the transform of $a^{|n|}$, so $|X|^2$ is that transform divided by $1-a^2$, and its synthesis integral at $n=0$ returns $a^{|0|} = 1$:

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}|X(e^{j\omega})|^2\,\mathrm{d}\omega &= \frac{1}{1-a^2}\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-a^2}{1-2a\cos\omega+a^2}\,\mathrm{d}\omega \\ &= \frac{1}{1-a^2}\cdot 1 = \frac{1}{1-\frac{1}{4}} = \tfrac{4}{3}.\end{aligned}$$

For the rectangular pulse with $N_1 = 2$, the time side is five samples of 1^2, so 5. On the frequency side, expand the squared modulus as a double sum and integrate term by term:

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}\Big|\sum_{n=-2}^{2}e^{-j\omega n}\Big|^2\mathrm{d}\omega &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=-2}^{2}\sum_{m=-2}^{2}e^{-j\omega(n-m)}\,\mathrm{d}\omega \\ &= \sum_{n=-2}^{2}\sum_{m=-2}^{2}\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-j\omega(n-m)}\,\mathrm{d}\omega = 5.\end{aligned}$$

For an integer $p \neq 0$, $\int_{-\pi}^{\pi}e^{-j\omega p}\mathrm{d}\omega = \big[e^{-j\omega p}/(-jp)\big]_{-\pi}^{\pi} = \frac{e^{-j\pi p}-e^{j\pi p}}{-jp} = \frac{2\sin(\pi p)}{p} = 0$; for $p=0$ the integral is 2π. So of the 25 terms only the 5 with $m=n$ survive, each contributing 1. Likewise $(0.8)^n u[n]$ has energy $\sum_{n\geq 0}(0.64)^n = 1/(1-0.64) = 2.78$.

Duality

There is no duality inside the discrete-time transform pair. The analysis equation is a sum over an integer and the synthesis equation an integral over a continuous variable, so no renaming turns one into the other. Two dualities do hold.

The discrete-time series. Both of its domains are discrete and both objects have period N, so the coefficients can be read as a sequence $a[n]$. Its own coefficients are $b_k = \frac{1}{N}\sum_{n=\langle N\rangle} a[n]e^{-jk(2\pi/N)n}$. Compare the synthesis equation of x at the index $-k$: $x[-k] = \sum_{n=\langle N\rangle} a[n]e^{jn(2\pi/N)(-k)} = \sum_{n=\langle N\rangle} a[n]e^{-jk(2\pi/N)n}$. The two sums are the same, so $b_k = \frac{1}{N}x[-k]$:

$$x[n] \longleftrightarrow a_k \qquad \Longrightarrow \qquad a[n] \longleftrightarrow \frac{1}{N}x[-k].$$

The impulse train with $N=8$ is the example. By Example 6.9 its coefficients are all $a_k = 1/8 = 0.125$. Read as a sequence, the constant $1/8$ has as its coefficients the train itself, scaled by $1/8$. Likewise, $x[n] = 1$ with $N=4$ has $a_0 = 1$ and $a_1 = a_2 = a_3 = 0$; the sequence $a[n]$ is then a unit sample every four indices, whose coefficients are $\frac{1}{4}x[-k] = \frac{1}{4}$ for every k.

The transform as a Fourier series. $X(e^{j\omega})$ is a 2π-periodic function of the continuous variable ω. In the analysis sum put $k = -n$. As n runs over all integers, so does k:

$$X(e^{j\omega}) = \sum_n x[n]\, e^{-j\omega n} = \sum_k x[-k]\, e^{jk\omega}.$$

This is a continuous-time Fourier series in the variable ω, with period 2π, fundamental frequency 1 and coefficients $a_k = x[-k]$: the spectrum of a sequence is a periodic signal whose series coefficients are the sequence itself, reversed. As a check, take the 2π-periodic square wave in ω equal to 1 for $|\omega| \le \pi/2$ and zero over the rest of the period. Its coefficients come from the analysis integral over one period. The function is zero outside $|\omega| \le \pi/2$, so the limits shrink, and for $k \neq 0$ the antiderivative of $e^{-jk\omega}$ is $e^{-jk\omega}/(-jk)$:

$$\begin{aligned} a_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{-jk\omega}\,\mathrm{d}\omega = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{-jk\omega}\,\mathrm{d}\omega = \frac{1}{2\pi}\left[\frac{e^{-jk\omega}}{-jk}\right]_{-\pi/2}^{\pi/2} \\ &= \frac{1}{2\pi}\cdot\frac{e^{jk\pi/2}-e^{-jk\pi/2}}{jk} = \frac{1}{2\pi}\cdot\frac{2j\sin(k\pi/2)}{jk} = \frac{\sin(k\pi/2)}{k\pi}. \end{aligned}$$

The fourth equality reverses the order of the two limit terms to absorb the minus sign, and the fifth uses $e^{j\theta} - e^{-j\theta} = 2j\sin\theta$. At $k=0$ the integrand is 1, so $a_0 = \frac{1}{2\pi}\cdot\pi = \frac{1}{2}$. The list is 0.5, $1/\pi = 0.3183$, 0, $-1/(3\pi) = -0.1061$ for $k = 0, 1, 2, 3$. The inverse transform of the same band is Example 6.6 with $W = \pi/2$: $x[n] = \sin(\pi n/2)/(\pi n)$ with $x[0] = \frac{1}{2}$, the same list of numbers. So $a_k = x[-k]$, and since this x is even, $a_k = x[k]$ as well. In the same way, $X(e^{j\omega}) = 3e^{-j2\omega}$ belongs to $x[n] = 3\delta[n-2]$, and its only nonzero series coefficient is $a_{-2} = x[2] = 3$.

6.5 Convolution and multiplication

CONVOLUTION

$$y[n] = x[n] * h[n] \;\longleftrightarrow\; Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

Here h is the impulse response of an LTI system and $H(e^{j\omega})$ its frequency response. One system seen in frequency: a convolution sum becomes one product at each frequency.

To prove it, put the convolution sum $y[n] = \sum_k x[k]h[n-k]$ into the analysis sum, swap the order of the two sums, and change the index of the inner sum with $m = n-k$. For fixed k, as n runs over all integers so does m:

$$\begin{aligned} Y(e^{j\omega}) &= \sum_n \Big[\sum_k x[k]h[n-k]\Big] e^{-j\omega n} = \sum_k x[k]\sum_n h[n-k]e^{-j\omega n} \\ &= \sum_k x[k]\sum_m h[m]e^{-j\omega(m+k)} = \Big[\sum_k x[k]e^{-j\omega k}\Big]\Big[\sum_m h[m]e^{-j\omega m}\Big] = X(e^{j\omega})H(e^{j\omega}). \end{aligned}$$

In the second line $e^{-j\omega(m+k)} = e^{-j\omega k}e^{-j\omega m}$, and the factor $e^{-j\omega k}$ does not depend on m, so it joins the outer sum. Swapping the sums is allowed when both sequences are absolutely summable. X and H repeat every 2π, so Y does too: if $X(e^{j\pi/2}) = 2$ and $H(e^{j\pi/2}) = 0.4$, then $Y(e^{j5\pi/2}) = Y(e^{j\pi/2}) = 0.8$.

A product of complex numbers multiplies the moduli and adds the angles, so the property gives two separate statements:

$$|Y(e^{j\omega})| = |X(e^{j\omega})|\,|H(e^{j\omega})|, \qquad \angle Y(e^{j\omega}) = \angle X(e^{j\omega}) + \angle H(e^{j\omega}).$$

⊗ NO BARS AROUND AN EQUATION

Writing $|Y = XH|$ puts modulus bars around an equation, which is not an operation on anything. Write the magnitude line and the phase line separately, and declare both pairs $x \leftrightarrow X$ and $h \leftrightarrow H$ before the conclusion.

EXAMPLE 6.13 – CONVOLVING TWO EXPONENTIALS

GIVEN $x[n] = a^n u[n]$ and $h[n] = b^n u[n]$, with $|a| < 1$, $|b| < 1$ and $a \neq b$.

FIND $y[n] = x[n] * h[n]$.

METHOD Use the convolution property because the required output is a time convolution. Multiply the transforms, then use partial fractions so that each term matches the one-sided exponential pair. Write $z = e^{-j\omega}$ as an algebraic variable. Then $Y = 1/((1-az)(1-bz))$ is a rational function of z, and setting $z = 1/a$ is algebra in z only, not a claim that $e^{-j\omega}$ takes that value.

SOLUTION By Example 6.3, $X = 1/(1-az)$ and $H = 1/(1-bz)$ with $z = e^{-j\omega}$, so the convolution property gives

$$Y = X\,H = \frac{1}{(1-az)(1-bz)} = \frac{A}{1-az} + \frac{B}{1-bz}.$$

Cover-up: to find A, cover the factor $(1-az)$ and evaluate the rest at $z = 1/a$, the value that makes that factor zero. To find B, cover $(1-bz)$ and evaluate at $z = 1/b$:

$$A = \left.\frac{1}{1-bz}\right|_{z=1/a} = \frac{1}{1-b/a} = \frac{a}{a-b},$$
$$B = \left.\frac{1}{1-az}\right|_{z=1/b} = \frac{1}{1-a/b} = \frac{b}{b-a} = -\frac{b}{a-b}.$$

Check by recombining: $A(1-bz) + B(1-az) = \frac{a-abz-b+abz}{a-b} = 1$, as required. Each fraction is the one-sided exponential pair, so inverting term by term:

$$y[n] = A\,a^n u[n] + B\,b^n u[n] = \frac{a\cdot a^n - b\cdot b^n}{a-b}\,u[n] = \frac{1}{a-b}\left[a^{n+1} - b^{n+1}\right]u[n].$$

CONDITION Both coefficients divide by $a-b$, so the route requires $a \neq b$. At $a = b$ the two factors are the same, and Section 6.6 derives the pair that replaces this expansion.

CHECK $a = \frac{1}{2}$, $b = \frac{1}{4}$: $a - b = \frac{1}{4}$, so $y[n] = 4\left[(\frac{1}{2})^{n+1} - (\frac{1}{4})^{n+1}\right]u[n]$, giving $y[0] = 4(\frac{1}{2} - \frac{1}{4}) = 1$, $y[1] = 4(\frac{1}{4} - \frac{1}{16}) = 0.75$, $y[2] = 4(\frac{1}{8} - \frac{1}{64}) = 0.4375$, $y[3] = 4(\frac{1}{16} - \frac{1}{256}) = 0.234375$. Direct convolution gives $y[0] = x[0]h[0] = 1$ and $y[1] = x[0]h[1] + x[1]h[0] = \frac{1}{4} + \frac{1}{2} = 0.75$. The first value must be $x[0]h[0] = 1$ for any two such sequences. The magnitude $|Y| = |X|\,|H|$ is largest at $\omega = 0$, where $e^{-j\omega} = 1$: $\frac{1}{1-\frac{1}{2}} \cdot \frac{1}{1-\frac{1}{4}} = 2 \cdot \frac{4}{3} = \frac{8}{3} = 2.6667$. It is smallest at $\omega = \pi$, where $e^{-j\pi} = -1$: $\frac{1}{1+\frac{1}{2}} \cdot \frac{1}{1+\frac{1}{4}} = \frac{2}{3} \cdot \frac{4}{5} = \frac{8}{15} = 0.5333$.

READING Every one-sided exponential carries its $u[n]$. Without the step, $(\frac{1}{2})^n$ is defined for negative n too, grows without bound as $n \to -\infty$, and has no transform.

EXAMPLE 6.14 — A CASCADE OF IDEAL LOW-PASS FILTERS

GIVEN H_1 is ideal low-pass with cutoff $\pi/2$ and H_2 is ideal low-pass with cutoff $\pi/4$, in cascade, with input $x[n] = \delta[n]$.

FIND The overall frequency response H and the output $y[n]$.

METHOD A cascade convolves the impulse responses, so the frequency responses multiply. Multiply the two bands, then invert the product with Example 6.6. Each band is 1 for $|\omega| \le W$ and 0 for $W < |\omega| \le \pi$: the band edge belongs to one branch only.

SOLUTION The product is 1 where both factors are 1, that is on $|\omega| \le \pi/4$, and 0 elsewhere in the period:

$$H(e^{j\omega}) = H_1 H_2 = \begin{cases} 1, & |\omega| \le \pi/4 \\ 0, & \pi/4 < |\omega| \le \pi. \end{cases}$$

The input $\delta[n]$ has $X = 1$, so $Y = H$ and Example 6.6 with $W = \pi/4$ gives

$$y[n] = \frac{\sin(\pi n/4)}{\pi n}, \qquad y[0] = \frac{\pi/4}{\pi} = \frac{1}{4}.$$

CHECK $y[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H \,\mathrm{d}\omega = \frac{\pi/2}{2\pi} = \frac{1}{4}$, the band width $\pi/2$ over 2π. The product commutes, so the order of the filters does not matter; the cascade keeps only the band both filters pass.

EXAMPLE 6.15 — A STEPPED SPECTRUM THROUGH A FILTER

GIVEN $X(e^{j\omega}) = 2$ for $|\omega| \le \pi/4$, 1 for $\pi/4 < |\omega| \le 3\pi/4$, and 0 for $3\pi/4 < |\omega| \le \pi$; H is ideal low-pass with cutoff $\pi/2$.

FIND $Y(e^{j\omega})$ and $y[n]$.

METHOD An LTI system gives $Y = XH$. Multiply the two spectra frequency by frequency, then write the result as a sum of ideal low-pass bands so that Example 6.6 inverts each one.

SOLUTION H is 1 for $|\omega| \le \pi/2$ and 0 for $\pi/2 < |\omega| \le \pi$. Multiplying band by band: for $|\omega| \le \pi/4$, $Y = 2 \cdot 1 = 2$; for $\pi/4 < |\omega| \le \pi/2$, $Y = 1 \cdot 1 = 1$; for $\pi/2 < |\omega| \le \pi$, $Y = X \cdot 0 = 0$. Now write Y as two stacked ideal bands, $Y = Y_1 + Y_2$, with $Y_1 = 1$ on $|\omega| \le \pi/2$ and $Y_2 = 1$ on $|\omega| \le \pi/4$. The two add to 2 on the inner band and to 1 on the outer band, as required. By linearity and Example 6.6 with $W = \pi/2$ and $W = \pi/4$:

$$\begin{aligned} y[n] &= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{j\omega n}\mathrm{d}\omega + \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} e^{j\omega n}\mathrm{d}\omega \\ &= \frac{\sin(\pi n/2)}{\pi n} + \frac{\sin(\pi n/4)}{\pi n}, \end{aligned}$$

with $y[0] = \frac{\pi/2}{\pi} + \frac{\pi/4}{\pi} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$.

CHECK $y[0]$ must be the area of one period of Y divided by 2π. That area is $2 \cdot \frac{\pi}{2} + 1 \cdot \frac{\pi}{2} = \frac{3\pi}{2}$ (height 2 over a band of width $\pi/2$, height 1 over two bands of width $\pi/4$), and $\frac{3\pi}{2}/2\pi = \frac{3}{4}$.

Multiplication is a periodic convolution

MULTIPLICATION

$$z[n] = x[n]y[n] \;\longleftrightarrow\; Z(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})\,Y\left(e^{j(\omega-\theta)}\right)\mathrm{d}\theta$$

The integral runs over one period and both factors are 2π-periodic. That operation is a **periodic convolution**. An integral over all θ would count the same overlap once in every period and diverge.

To prove it, put the product into the analysis sum and replace $x[n]$ by its synthesis integral, with θ as the integration variable so that it is not confused with ω. Then swap the sum and the integral and combine the exponentials, $e^{j\theta n}e^{-j\omega n} = e^{-j(\omega-\theta)n}$:

$$\begin{aligned} Z(e^{j\omega}) &= \sum_n x[n]y[n]e^{-j\omega n} = \sum_n \left[\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})e^{j\theta n}\,\mathrm{d}\theta\right] y[n]e^{-j\omega n} \\ &= \frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})\left[\sum_n y[n]e^{-j(\omega-\theta)n}\right]\mathrm{d}\theta = \frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})\,Y\left(e^{j(\omega-\theta)}\right)\mathrm{d}\theta. \end{aligned}$$

The bracket is the analysis sum of y at frequency $\omega-\theta$. Read the formula one frequency at a time: for a chosen ω, slide Y so that it is centred at ω, multiply it by X over one period, and divide the area of the product by 2π. That is one value of Z. Moving ω by 2π moves Y by one whole period, so the overlap does not change:

$$Z\left(e^{j(\omega+2\pi)}\right) = \frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y\left(e^{j(\omega-\theta+2\pi)}\right)\mathrm{d}\theta = \frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y\left(e^{j(\omega-\theta)}\right)\mathrm{d}\theta = Z(e^{j\omega}).$$

At $\omega = 0$ the synthesis equation gives a quick check: $\frac{1}{2\pi}\int_{2\pi} Z(e^{j\omega})\,\mathrm{d}\omega = z[0] = x[0]\,y[0]$. If $x[0] = 2$ and $y[0] = 0.5$, that integral is 1.

EXAMPLE 6.16 — SPECTRAL OVERLAP AT A BAND EDGE

GIVEN $x[n] = \dfrac{\sin(3\pi n/4)}{\pi n}$ and $y[n] = \dfrac{\sin(\pi n/2)}{\pi n}$. By Example 6.6 these are bands of height 1 with half-widths $3\pi/4$ and $\pi/2$, repeated every 2π.

FIND $Z(e^{j\omega})$ for $z[n] = x[n]y[n]$.

METHOD First convolve the two bands as functions on the line. Then add the copies centred at multiples of 2π that reach into the period, because the periodic convolution counts them.

SOLUTION The integrand $X(\theta)Y(\omega-\theta)$ is 1 where both factors are 1 and 0 elsewhere, so the integral is the length of the overlap of $|\theta| \le 3\pi/4$ with $\omega - \pi/2 \le \theta \le \omega + \pi/2$. For $|\omega| \le \pi/4$ the second interval lies inside the first, so the overlap is its full width π and the value is $\frac{1}{2\pi}\cdot\pi = \frac{1}{2}$. For $\pi/4 \le |\omega| \le 5\pi/4$ the overlap runs from $|\omega| - \pi/2$ to $3\pi/4$, of length $5\pi/4 - |\omega|$, so the value is $(5\pi/4 - |\omega|)/(2\pi)$, falling to zero at $|\omega| = 5\pi/4$. The result on the line is a trapezoid of height $\frac{1}{2}$, flat on $|\omega| \le \pi/4$. Since $5\pi/4 > \pi$, it reaches beyond one period, and near $\omega = \pm\pi$ the copy centred at $\pm 2\pi$ overlaps it. The values add:

$$Z(e^{j0}) = \tfrac{1}{2}, \qquad Z(e^{j3\pi/4}) = \frac{5\pi/4 - 3\pi/4}{2\pi} = \tfrac{1}{4},$$
$$Z(e^{j\pi}) = \underbrace{\frac{5\pi/4 - \pi}{2\pi}}_{\text{own copy}} + \underbrace{\frac{5\pi/4 - \pi}{2\pi}}_{\text{next copy}} = \tfrac{1}{8} + \tfrac{1}{8} = \tfrac{1}{4}.$$

The copy centred at 2π is at the same distance π from $\omega = \pi$ as the own copy, so it gives the same $\frac{1}{8}$. Between $3\pi/4$ and π one copy falls as the other rises, so Z falls in a straight line from $\frac{1}{2}$ at $\pi/4$ to $\frac{1}{4}$ at $3\pi/4$ and then stays at $\frac{1}{4}$ across $\pm\pi$.

CHECK

By Example 6.6, $x[0] = \frac{3\pi/4}{\pi} = \frac{3}{4}$ and $y[0] = \frac{\pi/2}{\pi} = \frac{1}{2}$, so $z[0] = \frac{3}{8} = 0.375$. On the frequency side, one period of Z collects the whole area of one trapezoid, because the parts that spill past $\pm\pi$ are exactly the parts that the copies bring in. That area is the height $\frac{1}{2}$ times the mean of the top width $\pi/2$ and the base width $5\pi/2$: $\frac{1}{2} \cdot \frac{3\pi}{2} = \frac{3\pi}{4}$, and $\frac{3\pi/4}{2\pi} = \frac{3}{8}$, as required.

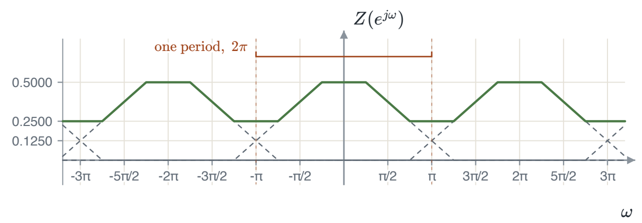

FIGURE 6.25 The dashed trapezoids are the copies; the solid curve is their sum. At $\omega = \pi$ the value doubles from 0.125 to 0.25.

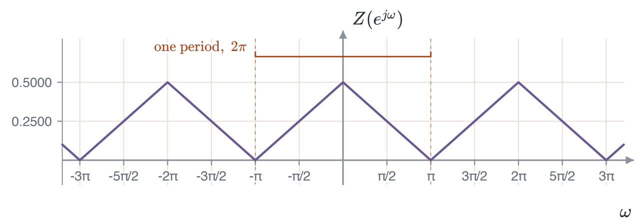

FIGURE 6.26 Two equal bands of half-width $\pi/2$: each copy is a triangle of peak $\frac{1}{2}$ that ends exactly at $\pm\pi$, so the copies do not overlap.

When do the copies overlap? On the line, the convolution of two bands of half-widths W_X and W_Y is nonzero for $|\omega| < W_X + W_Y$. The copies are centred 2π apart, so two neighbours meet exactly when each reaches past the midpoint between them:

$$\text{copies overlap} \iff W_X + W_Y > \pi.$$

If each copy ends inside its own period, periodic and ordinary convolution give the same values on that period. Bands of half-widths $\pi/4$ and $\pi/2$ reach only $3\pi/4 < \pi$, so their copies stay apart; bands of $3\pi/4$ and $\pi/2$ reach $5\pi/4 > \pi$ and overlap, as in Example 6.16. If Y is narrowed to half-width $\pi/4$ while X keeps $3\pi/4$, then at $\omega = 0$ the band of Y lies inside X, the overlap is $\pi/2$ wide and $Z(e^{j0}) = \frac{\pi/2}{2\pi} = \frac{1}{4}$.

⚠ THIS OVERLAP IS NOT ALIASING

This overlap is between copies of a periodic spectrum, and it can happen for any product of sequences. Chapter 7 studies a different overlap, between replicas produced by sampling. Only the sampling overlap is called aliasing.

EXAMPLE 6.17 — MULTIPLYING BY A COSINE

GIVEN

$X(e^{j\omega}) = 1$ for $|\omega| \le \pi/4$ and 0 for $\pi/4 < |\omega| \le \pi$, repeated every 2π, and $y[n] = \cos(\omega_0 n)$ with $\omega_0 = \pi/3$.

FIND $Z(e^{j\omega})$ for $z[n] = x[n]y[n]$, the band edges, and the range of ω_0 for which the bands stay apart.

METHOD Write Y as its impulse train, $\pi\sum_l[\delta(\omega-\omega_0-2\pi l)+\delta(\omega+\omega_0-2\pi l)]$. The periodic convolution integrates over one period, so choosing $-\pi<\theta\le\pi$ leaves only the two impulses with $l=0$, at $\theta=\pm\pi/3$. Each impulse places a copy of X.

SOLUTION Periodic convolution commutes, so put the impulse train first in the multiplication property and integrate over $-\pi<\theta\le\pi$. The sifting property evaluates X at the two impulse positions:

$$\begin{aligned}
Z(e^{j\omega}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\theta})\,X\left(e^{j(\omega-\theta)}\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \pi\left[\delta(\theta-\omega_0)+\delta(\theta+\omega_0)\right]X\left(e^{j(\omega-\theta)}\right)\mathrm{d}\theta \\
&= \frac{\pi}{2\pi}\left[X\left(e^{j(\omega-\omega_0)}\right)+X\left(e^{j(\omega+\omega_0)}\right)\right] \\
&= \tfrac{1}{2}X\left(e^{j(\omega-\omega_0)}\right)+\tfrac{1}{2}X\left(e^{j(\omega+\omega_0)}\right).
\end{aligned}$$

The height $\frac{1}{2}$ of each copy comes from $\frac{1}{2\pi}\cdot\pi$. The upper band runs from $\frac{\pi}{3}-\frac{\pi}{4}=\frac{\pi}{12}$ to $\frac{\pi}{3}+\frac{\pi}{4}=\frac{7\pi}{12}$, and the mirror copy covers $-7\pi/12\le\omega\le-\pi/12$.

RANGE The upper band is $[\omega_0-\pi/4,\ \omega_0+\pi/4]$ and its mirror is $[-\omega_0-\pi/4,\ -\omega_0+\pi/4]$. They stay apart at $\omega=0$ when $\omega_0-\pi/4\ge-\omega_0+\pi/4$, that is $\omega_0\ge\pi/4$. The mirror copy of the next period is centred at $2\pi-\omega_0$, so it starts at $2\pi-\omega_0-\pi/4$. It stays clear of the upper band when $2\pi-\omega_0-\pi/4\ge\omega_0+\pi/4$, that is $\omega_0\le3\pi/4$. So the bands stay apart for $\pi/4\le\omega_0\le3\pi/4$. A smaller ω_0 makes them overlap at $\omega=0$, a larger one at $\omega=\pm\pi$.

CHECK $z[0]=x[0]\cos 0=\frac{\pi/4}{\pi}\cdot 1=\frac{1}{4}$. From Z: two bands of height $\frac{1}{2}$ and width $\frac{\pi}{2}$ have area $\frac{\pi}{2}$ per period, and $\frac{\pi/2}{2\pi}=\frac{1}{4}$.

Seeing a cosine through a window

A measured record is always finite. Keep L samples of a cosine: $x[n]=\cos(\omega_0 n)\,w[n]$, where $w[n]=1$ for $0\le n\le L-1$ is the rectangular window of Section 6.1, with transform $W(e^{j\omega})$. By Euler's relation, $x[n]=\frac{1}{2}e^{j\omega_0 n}w[n]+\frac{1}{2}e^{-j\omega_0 n}w[n]$, and the frequency-shift property moves W to $\pm\omega_0$:

$$X(e^{j\omega})=\tfrac{1}{2}\,W\left(e^{j(\omega-\omega_0)}\right)+\tfrac{1}{2}\,W\left(e^{j(\omega+\omega_0)}\right),\qquad |W(e^{j\omega})|=\left|\frac{\sin(\omega L/2)}{\sin(\omega/2)}\right|.$$

The multiplication property gives the same result, because the cosine's impulses of weight π sift out two copies of W of height $\frac{1}{2\pi}\cdot\pi=\frac{1}{2}$. The endless cosine has two lines at $\pm\omega_0$. The kept part has two copies of W instead. Each line becomes a main lobe, whose first zeros are at $\pm2\pi/L$ from its centre, so it is $4\pi/L$ wide. The side lobes of W spread a little of the cosine into every other frequency. This spreading is called **leakage**. The main lobe is $\pi/4$ wide for $L=16$ and half as wide, $\pi/8$, for $L=32$. Near its centre each copy has height $\frac{1}{2}L$, so $\frac{2}{L}|X|$ is close to 1 at $\pm\omega_0$ when the copies are well apart.

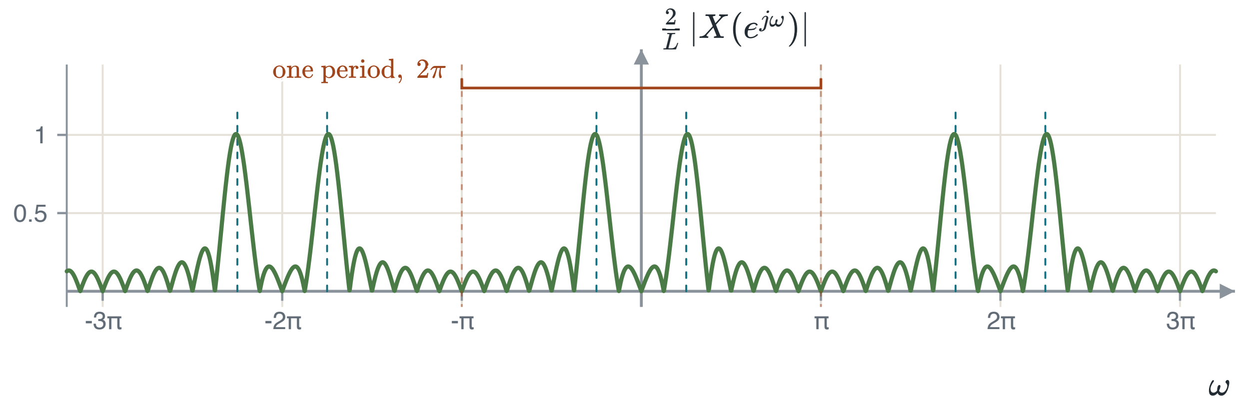

FIGURE 6.27 $\cos(\pi n/4)$ kept for $L = 16$ samples. The dashed lines mark $\pm\pi/4$ and their copies, where the endless cosine has its impulses. Each line has become a main lobe $\pi/4$ wide with side lobes around it.

A softer window trades width for leakage. The **Hann window** is $w[n] = \sin^2(\pi n/L)$ for $0 \le n \le L-1$ and 0 elsewhere. By $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$ it equals $\frac{1}{2} - \frac{1}{2}\cos(2\pi n/L)$ on the window, which is the rectangular window times a constant plus a cosine. The same Euler and frequency-shift steps, with $\omega_0 = 2\pi/L$, give

$$W_{\mathrm{H}}(e^{j\omega}) = \tfrac{1}{2}W(e^{j\omega}) - \tfrac{1}{4}W\big(e^{j(\omega-2\pi/L)}\big) - \tfrac{1}{4}W\big(e^{j(\omega+2\pi/L)}\big).$$

W is zero at every nonzero multiple of $2\pi/L$ inside the period. At $\omega = 2\pi/L$ the three terms are $\frac{1}{2}W$ at $2\pi/L$, which is zero, $-\frac{1}{4}W$ at 0, which is $-\frac{1}{4}L$, and $-\frac{1}{4}W$ at $4\pi/L$, which is zero. So $|W_{\mathrm{H}}| = L/4$ there: the Hann window has no zero at $2\pi/L$. At $\omega = 4\pi/L$ the three arguments are $4\pi/L$, $2\pi/L$ and $6\pi/L$, all zeros of W, so $W_{\mathrm{H}} = 0$. The first zero moves out to $4\pi/L$ and the main lobe doubles to $8\pi/L$. In return the three shifted copies partly cancel away from the main lobe, and the highest side lobe drops from about -13 dB for the rectangular window to about -31 dB for the Hann window, measured against the peak.

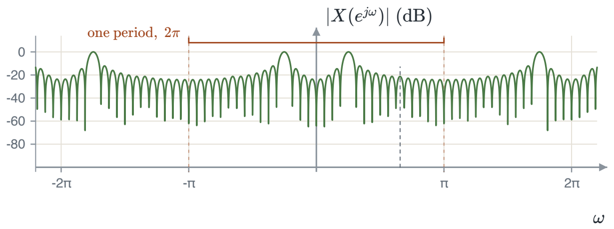

FIGURE 6.28 Rectangular window, $L = 32$. The weak cosine at the dashed line $21\pi/32$ lies below the side lobes of the strong one.

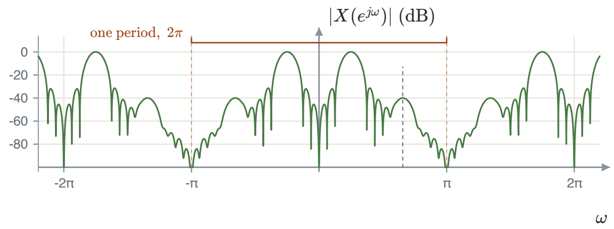

FIGURE 6.29 Hann window, $L = 32$. The main lobes are wider, the side lobes much lower, and the weak cosine shows as its own peak.

The figures use $x[n] = \cos(\pi n/4) + 0.01\cos(21\pi n/32)$ kept for $L = 32$ samples, in dB against the strong peak. The second cosine is $20\log_{10} 0.01 = -40$ dB weaker than the first. Under the rectangular window the side lobes of the strong cosine near $\omega_2 = 21\pi/32$ are near -21 dB, far above -40 dB, and hide the weak peak. The Hann side lobes there are much lower, and the weak cosine appears. If the weak cosine is made ten times weaker still, -60 dB, the Hann side lobes near ω_2, at about -54 dB, hide it again.

Spectrum over time

A long sequence whose content changes, such as music or speech, is better described by a sequence of spectra than by one. Slide a window of L samples along the sequence and take the transform of each piece:

$$X_m(e^{j\omega}) = \sum_{n=m}^{m+L-1} x[n]\, w[n-m]\, e^{-j\omega n}.$$

This is the analysis sum of the windowed piece that starts at $n = m$. Drawing $|X_m(e^{j\omega})|$ as a column of colours for each start m, side by side, gives a **spectrogram**: time runs to the right and frequency upward. For a real x, $|X_m|$ is even in ω and repeats every 2π, so $0 \le \omega \le \pi$ is enough; this is the one kind of figure in the chapter drawn over less than a period. The window sets a trade-off. A Hann window of $L = 32$ has a main lobe $8\pi/32 = \pi/4$ wide. With $L = 64$ the main lobe is $\pi/8$ and the bands are narrower, but each window spans a change of note in more columns, so the map blurs more in time.

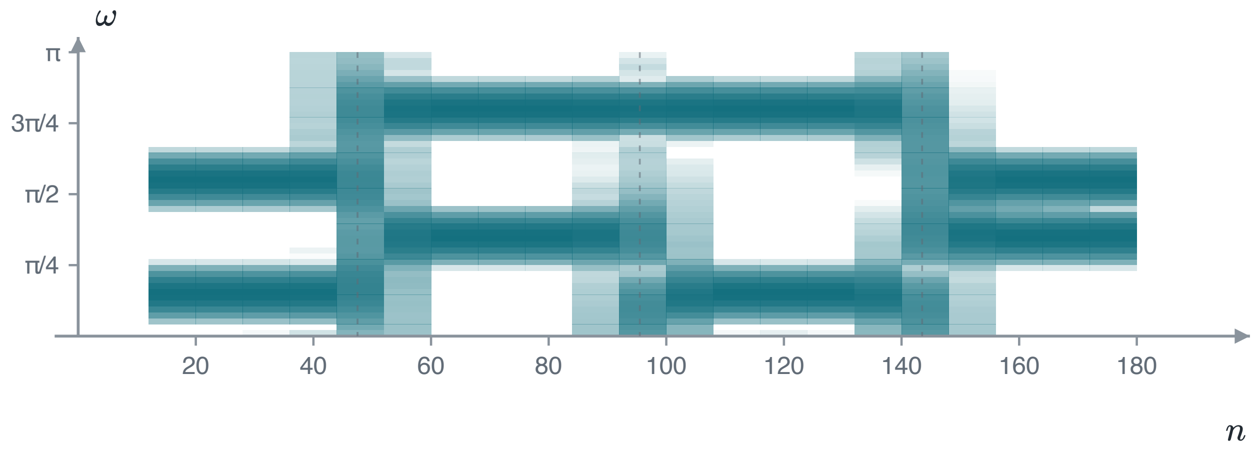

FIGURE 6.30 Spectrogram of four notes of 48 samples, each a pair of tones: 0.15π and 0.55π, then 0.35π and 0.80π, then 0.15π and 0.80π, then 0.35π and 0.55π. Hann window, $L = 32$, a new column every 8 samples; stronger colour is a larger magnitude, over a range of 30 dB. Each note shows as two bands, and the columns whose window spans two notes are blurred.

6.6 Difference equations

Transform both sides of a linear constant-coefficient difference equation. Linearity acts term by term, and the time-shift property turns each $y[n-k]$ into $e^{-j\omega k}Y(e^{j\omega})$ and each $x[n-k]$ into $e^{-j\omega k}X(e^{j\omega})$. The transforms Y and X do not depend on k, so they come out of the sums:

FREQUENCY RESPONSE OF A DIFFERENCE EQUATION

$$\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k]$$

$$Y(e^{j\omega}) \sum_{k=0}^{N} a_k e^{-j\omega k} = X(e^{j\omega}) \sum_{k=0}^{M} b_k e^{-j\omega k}$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}$$

The last line divides both sides by $X(e^{j\omega})$ and by the denominator sum. H is a ratio of polynomials in $e^{-j\omega}$. The impulse response of the recursion is now available without running it: factor the denominator, split into partial fractions, and invert each piece with the exponential pair. $H(e^{j\omega})$ exists only for a stable system, $\sum_n |h[n]| < \infty$; in the examples below every factor has $|a| < 1$. For instance, $y[n] - \frac{1}{2}y[n-1] = x[n] + x[n-1]$ has $H(e^{j\omega}) = (1 + e^{-j\omega})/(1 - \frac{1}{2}e^{-j\omega})$, and at $\omega = \pi$, $e^{-j\pi} = -1$ makes the numerator zero, so $H(e^{j\pi}) = 0$.

EXAMPLE 6.18 — A SECOND-ORDER SYSTEM

GIVEN The causal system $y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]$.

FIND The frequency response and the impulse response.

METHOD The equation has constant coefficients, so the time-shift property turns it into algebra. Read the coefficients into the frequency-response ratio, factor the denominator, expand into partial fractions, and invert each term with the one-sided exponential pair.

SOLUTION Here $a_0 = 1$, $a_1 = -\frac{3}{4}$, $a_2 = \frac{1}{8}$ and $b_0 = 2$. Transforming term by term:

$$Y - \tfrac{3}{4}e^{-j\omega}Y + \tfrac{1}{8}e^{-j2\omega}Y = 2X \quad\Longrightarrow\quad H(e^{j\omega}) = \frac{Y}{X} = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-j2\omega}}.$$

Write $z = e^{-j\omega}$ as an algebraic variable and factor the quadratic $1 - \frac{3}{4}z + \frac{1}{8}z^2$. Two factors $(1-\alpha z)(1-\beta z)$ expand to $1 - (\alpha+\beta)z + \alpha\beta z^2$, so we need $\alpha + \beta = \frac{3}{4}$ and $\alpha\beta = \frac{1}{8}$; $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{4}$ satisfy both, since $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ and $\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$. Hence

$$H = \frac{2}{(1 - \frac{1}{2}z)(1 - \frac{1}{4}z)} = \frac{A}{1 - \frac{1}{2}z} + \frac{B}{1 - \frac{1}{4}z}.$$

Cover-up: cover $(1 - \frac{1}{2}z)$ and set $z = 2$, then cover $(1 - \frac{1}{4}z)$ and set $z = 4$:

$$A = \left.\frac{2}{1 - \frac{1}{4}z}\right|_{z=2} = \frac{2}{1 - \frac{1}{2}} = 4, \qquad B = \left.\frac{2}{1 - \frac{1}{2}z}\right|_{z=4} = \frac{2}{1 - 2} = -2.$$

Recombining, $4(1 - \frac{1}{4}z) - 2(1 - \frac{1}{2}z) = 4 - z - 2 + z = 2$, which matches the numerator. Each fraction is the pair $a^n u[n] \leftrightarrow 1/(1 - ae^{-j\omega})$ with $|a| < 1$, so

$$h[n] = 4\left(\tfrac{1}{2}\right)^n u[n] - 2\left(\tfrac{1}{4}\right)^n u[n].$$

CHECK $h[0] = 4 - 2 = 2$, $h[1] = 4 \cdot \frac{1}{2} - 2 \cdot \frac{1}{4} = 2 - 0.5 = 1.5$, $h[2] = 4 \cdot \frac{1}{4} - 2 \cdot \frac{1}{16} = 1 - 0.125 = 0.875$. Substituting into the equation with $x = \delta$, which is zero for $n \geq 1$: $h[1] - \frac{3}{4}h[0] = 1.5 - 1.5 = 0$ and $h[2] - \frac{3}{4}h[1] + \frac{1}{8}h[0] = 0.875 - 1.125 + 0.25 = 0$, as required.

READING Both factors have $|a| < 1$, so both terms decay and the system is stable. At $\omega = 0$, $z = 1$ and $|H| = 2/(1 - \frac{3}{4} + \frac{1}{8}) = 2/\frac{3}{8} = 5.3333$; at $\omega = \pi$, $z = -1$ and $|H| = 2/(1 + \frac{3}{4} + \frac{1}{8}) = 2/\frac{15}{8} = 1.0667$. This recursion is a low-pass filter. The same factoring for $y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = x[n]$ gives $\frac{1}{2}$ and $\frac{1}{3}$, since $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ and $\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$, so its $h[n]$ is made of $(\frac{1}{2})^n$ and $(\frac{1}{3})^n$.

The repeated-pole pair

Example 6.13 excluded $a = b$ because its partial fractions divide by $a - b$. When $a = b$ the factors coincide and need a separate pair. Derive it from the geometric pair of Example 6.3, $\sum_{n=0}^{\infty} a^n e^{-j\omega n} = 1/(1 - ae^{-j\omega})$, by differentiating both sides with respect to a while holding ω fixed. On the left, differentiate term by term: $\frac{\mathrm{d}}{\mathrm{d}a}a^n = na^{n-1}$, and the $n = 0$ term is the constant 1, whose derivative is 0, so the sum starts at $n = 1$. On the right, use the chain rule on $(1 - ae^{-j\omega})^{-1}$:

$$\sum_{n=1}^{\infty} n\,a^{n-1}e^{-j\omega n} = \frac{\mathrm{d}}{\mathrm{d}a}\left(1 - ae^{-j\omega}\right)^{-1} = -\left(1 - ae^{-j\omega}\right)^{-2} \cdot \left(-e^{-j\omega}\right) = \frac{e^{-j\omega}}{\left(1 - ae^{-j\omega}\right)^2}.$$

Now put $m = n - 1$ on the left, so $n = m + 1$ and m runs from 0 to ∞. The exponential splits as $e^{-j\omega(m+1)} = e^{-j\omega}e^{-j\omega m}$, and the factor $e^{-j\omega}$ leaves the sum and cancels the same factor on the right:

$$\begin{aligned} &\sum_{m=0}^{\infty}(m+1)\,a^m e^{-j\omega(m+1)} = e^{-j\omega}\sum_{m=0}^{\infty}(m+1)\,a^m e^{-j\omega m} = \frac{e^{-j\omega}}{\left(1 - ae^{-j\omega}\right)^2} \\ \Longrightarrow \quad &\sum_{m=0}^{\infty}(m+1)\,a^m e^{-j\omega m} = \frac{1}{\left(1 - ae^{-j\omega}\right)^2}. \end{aligned}$$

REPEATED POLE

$$(n+1)a^n u[n] \;\longleftrightarrow\; \frac{1}{\left(1 - ae^{-j\omega}\right)^2}, \qquad |a| < 1$$

The left-hand sum is the analysis sum of $(n+1)a^n u[n]$, with m renamed n. Differentiating a convergent geometric series term by term is allowed for $|a| < 1$, which is why the condition is kept. The exponent 2 comes from one derivative of $(1 - ae^{-j\omega})^{-1}$, and the minus sign stays because differentiating in a does not change it. One number settles any doubt: at $\omega = 0$ with $a = \frac{1}{4}$ the value is $(1 - \frac{1}{4})^{-2} = \frac{16}{9} = 1.7778 = \sum_{n \geq 0}(n+1)(\frac{1}{4})^n$, while a plus sign would give $(\frac{5}{4})^{-2} = 0.6400$. With $a = \frac{1}{2}$ the value at $\omega = 0$ is $1/(1 - \frac{1}{2})^2 = 4$. With $a = -\frac{1}{2}$ the magnitude is largest at $\omega = \pi$, where $1 + \frac{1}{2}e^{-j\pi} = \frac{1}{2}$ is smallest and $|X| = 4$.

The pair also follows from differentiation in frequency. Split $(n+1)a^n u[n] = n\,a^n u[n] + a^n u[n]$ and add the two known transforms over the common denominator:

$$\frac{ae^{-j\omega}}{\left(1 - ae^{-j\omega}\right)^2} + \frac{1}{1 - ae^{-j\omega}} = \frac{ae^{-j\omega} + \left(1 - ae^{-j\omega}\right)}{\left(1 - ae^{-j\omega}\right)^2} = \frac{1}{\left(1 - ae^{-j\omega}\right)^2}.$$

Differentiating $r-1$ times in a instead of once gives the pole of order r. The left side becomes $\sum_n n(n-1)\cdots(n-r+2)\,a^{n-r+1}e^{-j\omega n}$ and the right side $(r-1)!\,e^{-j\omega(r-1)}/(1-ae^{-j\omega})^r$. Shifting the index by $r-1$, cancelling $e^{-j\omega(r-1)}$ and dividing by $(r-1)!$:

$$\frac{(n+r-1)!}{n!\,(r-1)!}\,a^n u[n] \;\longleftrightarrow\; \frac{1}{\left(1-ae^{-j\omega}\right)^r}, \qquad |a|<1.$$

For $r=2$ the coefficient is $(n+1)!/n! = n+1$, the repeated-pole pair. Two identical first-order stages $y[n] = a\,y[n-1] + x[n]$ in cascade have this impulse response $(n+1)a^n u[n]$.

EXAMPLE 6.19 – THE OUTPUT OF THAT SYSTEM

GIVEN The system of Example 6.18 driven by $x[n] = \left(\frac{1}{4}\right)^n u[n]$.

FIND $y[n]$.

METHOD Use $Y = XH$ and partial fractions, because the output transform is rational and each fraction can be inverted with a known pair. The input factor coincides with one factor of the system, so the expansion needs a simple and a squared term for it.

SOLUTION With $z = e^{-j\omega}$, the input transform is $X = 1/(1-\frac{1}{4}z)$ by Example 6.3, and H is the factored form of Example 6.18. Multiply:

$$\begin{aligned} Y = X\,H &= \frac{1}{1-\frac{1}{4}z}\cdot\frac{2}{\left(1-\frac{1}{2}z\right)\left(1-\frac{1}{4}z\right)} = \frac{2}{\left(1-\frac{1}{2}z\right)\left(1-\frac{1}{4}z\right)^2} \\ &= \frac{A}{1-\frac{1}{4}z} + \frac{B}{\left(1-\frac{1}{4}z\right)^2} + \frac{C}{1-\frac{1}{2}z}. \end{aligned}$$

Cover-up reaches the simple factor and the highest power of the repeated factor. Cover $\left(1-\frac{1}{2}z\right)$ and set $z=2$:

$$C = \left.\frac{2}{\left(1-\frac{1}{4}z\right)^2}\right|_{z=2} = \frac{2}{(1-\frac{1}{2})^2} = \frac{2}{\frac{1}{4}} = 8.$$

Cover $(1-\frac{1}{4}z)^2$ and set $z=4$:

$$B = \left.\frac{2}{1-\frac{1}{2}z}\right|_{z=4} = \frac{2}{1-2} = -2.$$

Cover-up cannot reach A, so use one more value of z. Multiply the identity through by the full denominator and set $z=0$; every factor becomes 1, so $2 = A+B+C = A-2+8$ and $A=-4$. Recombine as a check: $-4(1-\frac{1}{4}z)(1-\frac{1}{2}z) - 2(1-\frac{1}{2}z) + 8(1-\frac{1}{4}z)^2$ expands to $(-4+3z-\frac{1}{2}z^2) + (-2+z) + (8-4z+\frac{1}{2}z^2) = 2$, the numerator. Invert term by term. The first and third fractions are the one-sided exponential pair; the middle one is the repeated-pole pair with $a=\frac{1}{4}$:

$$y[n] = -4\left(\tfrac{1}{4}\right)^n u[n] - 2(n+1)\left(\tfrac{1}{4}\right)^n u[n] + 8\left(\tfrac{1}{2}\right)^n u[n].$$

CHECK $y[0] = -4-2+8 = 2$, $y[1] = -4\cdot\frac{1}{4} - 2\cdot 2\cdot\frac{1}{4} + 8\cdot\frac{1}{2} = -1-1+4 = 2$, $y[2] = -\frac{4}{16} - \frac{6}{16} + 2 = 1.375$, $y[3] = -\frac{4}{64} - \frac{8}{64} + 1 = 0.8125$. Direct convolution gives

$y[0] = h[0]x[0] = 2 \cdot 1 = 2$ and $y[1] = h[0]x[1] + h[1]x[0] = 2 \cdot \frac{1}{4} + 1.5 \cdot 1 = 2$, the same values.

READING After a few samples only $8(\frac{1}{2})^n$ is left: the slower factor $\frac{1}{2}$ sets the decay. $|Y|$ is largest at $\omega = 0$, $2/(\frac{1}{2} \cdot (\frac{3}{4})^2) = 7.1111$, and smallest at $\omega = \pi$, $2/(\frac{3}{2} \cdot (\frac{5}{4})^2) = 0.8533$. With the input $(\frac{1}{3})^n u[n]$ instead, $\frac{1}{3}$ differs from both system factors, and Y splits into three simple terms.

EXAMPLE 6.20 — AN ECHO AND ITS REMOVAL

GIVEN An echo $y[n] = x[n] + \alpha\, x[n-D]$: a copy scaled by α, with $0 < \alpha < 1$, arrives D samples later.

FIND The frequency response and its extremes, a recursion that removes the echo, and the condition for that recursion to be stable.

METHOD Read H off the equation with the time-shift property. For the inverse, write a recursion whose response is $1/H$ and expand $1/H$ as a geometric series to find its impulse response.

SOLUTION By linearity and the time shift,

$$H(e^{j\omega}) = 1 + \alpha e^{-j\omega D}.$$

Its squared modulus is the sum of the squared real and imaginary parts:

$$\begin{aligned} |H(e^{j\omega})|^2 &= (1 + \alpha \cos \omega D)^2 + (\alpha \sin \omega D)^2 \\ &= 1 + 2\alpha \cos \omega D + \alpha^2. \end{aligned}$$

It is largest where $\cos \omega D = 1$, giving $(1+\alpha)^2$, and smallest where $\cos \omega D = -1$, giving $(1-\alpha)^2$. So $|H|$ swings between $1-\alpha$ and $1+\alpha$. The cosine $\cos \omega D$ repeats every $2\pi/D$, so the swing happens D times in each period: the response is a comb. To remove the echo, run

$$w[n] = y[n] - \alpha\, w[n-D].$$

Its equation $w[n] + \alpha w[n-D] = y[n]$ transforms to $(1 + \alpha e^{-j\omega D})W = Y$, so its response is $1/(1 + \alpha e^{-j\omega D}) = 1/H$. The cascade has response $H \cdot \frac{1}{H} = 1$, so $W = X$ and $w[n] = x[n]$. For the impulse response of the inverse, use the geometric series with ratio $r = -\alpha e^{-j\omega D}$, of modulus $|\alpha|$:

$$\frac{1}{1 + \alpha e^{-j\omega D}} = \frac{1}{1-r} = \sum_{k=0}^{\infty} r^k = \sum_{k=0}^{\infty} (-\alpha)^k e^{-j\omega k D}, \qquad |\alpha| < 1.$$

By the shift pair $\delta[n - kD] \leftrightarrow e^{-j\omega k D}$, the impulse response is $(-\alpha)^k$ at $n = kD$, $k \geq 0$, and zero elsewhere.

STABILITY $\sum_n |h[n]| = \sum_{k \geq 0} |\alpha|^k = 1/(1-|\alpha|)$ is finite only for $|\alpha| < 1$. For $|\alpha| \geq 1$ the terms $(-\alpha)^k$ do not decay and the recursion is unstable.

NUMBERS $\alpha = \frac{1}{2}$: $|H|$ between 0.5 and 1.5, and the inverse between $1/1.5 = 2/3$ and $1/0.5 = 2$. $\alpha = 0.8$: $|H|$ between 0.2 and 1.8, and the inverse between $1/1.8 = 0.5556$ and

$1/0.2 = 5.$

CHECK At every ω the two magnitudes multiply to 1. At 8000 samples per second a delay of $D = 2000$ samples is 0.25 s, an echo that can be heard; the recursion takes it out again.

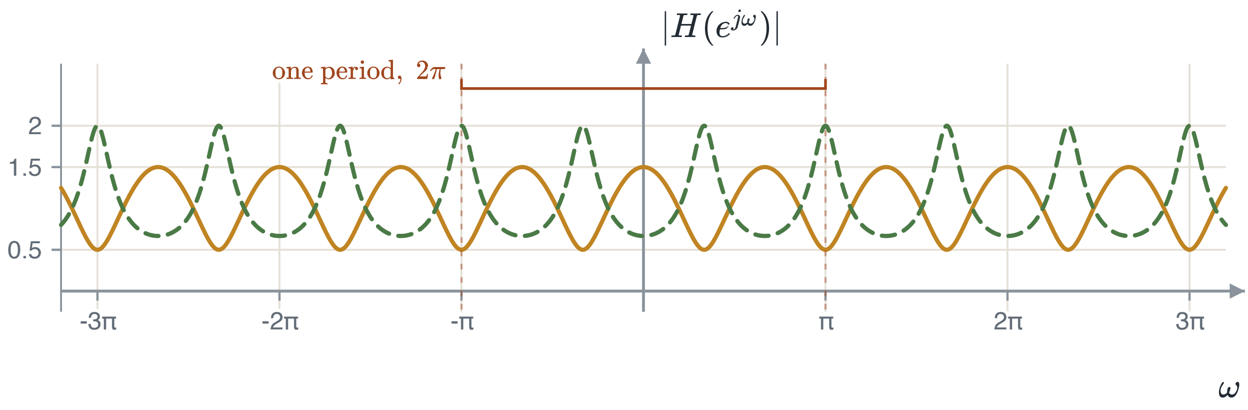

FIGURE 6.31 For $\alpha = \frac{1}{2}$ and $D = 3$: the echo response $|1 + \alpha e^{-j\omega D}|$ (solid) swings between 0.5 and 1.5 three times in each period; the response of the recursion that removes it (dashed) is its reciprocal, between $2/3$ and 2.

6.7 Summary

The two tables collect the properties and the pairs of this chapter. In every row, $x[n] \leftrightarrow X(e^{j\omega})$, each spectrum repeats every 2π, and each sum over k runs over all integers.

TABLE 6.1 Properties of the discrete-time Fourier transform.

PROPERTY	PAIR
Linearity	$a\,x_1[n] + b\,x_2[n] \leftrightarrow a\,X_1(e^{j\omega}) + b\,X_2(e^{j\omega})$
Time shift	$x[n-n_0] \leftrightarrow e^{-j\omega n_0}X(e^{j\omega})$
Frequency shift	$e^{j\omega_0 n}x[n] \leftrightarrow X(e^{j(\omega-\omega_0)})$
Conjugation	$x^*[n] \leftrightarrow X^*(e^{-j\omega})$
Time reversal	$x[-n] \leftrightarrow X(e^{-j\omega})$
Time expansion	$x_{(k)}[n] \leftrightarrow X(e^{jk\omega})$
Convolution	$x[n] * h[n] \leftrightarrow X(e^{j\omega})H(e^{j\omega})$
Multiplication	$x[n]y[n] \leftrightarrow \frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})\mathrm{d}\theta$
Periodicity	$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$ for every sequence
First difference	$x[n] - x[n-1] \leftrightarrow (1-e^{-j\omega})X(e^{j\omega})$
Accumulation	$\sum_{m=-\infty}^{n} x[m] \leftrightarrow \frac{X(e^{j\omega})}{1-e^{-j\omega}} + \pi X(e^{j0})\sum_k \delta(\omega-2\pi k)$
Differentiation in frequency	$n\,x[n] \leftrightarrow j\,\mathrm{d}X(e^{j\omega})/\mathrm{d}\omega$
Real sequence	$X(e^{-j\omega}) = X^*(e^{j\omega})$: $\lvert X\rvert$ even, $\angle X$ odd
Real and even	$X(e^{j\omega})$ real and even
Real and odd	$X(e^{j\omega})$ purely imaginary and odd
Even and odd parts	$\mathcal{E}\mathrm{v}\{x\} \leftrightarrow \mathrm{Re}\{X\}$, $\mathcal{O}\mathrm{dd}\{x\} \leftrightarrow j\,\mathrm{Im}\{X\}$, for real x
Parseval	$\sum_n \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{2\pi} \lvert X(e^{j\omega})\rvert^2 \mathrm{d}\omega$

TABLE 6.2 Discrete-time Fourier transform pairs.

SEQUENCE	TRANSFORM
$\delta[n]$	1
$\delta[n-n_0]$	$e^{-j\omega n_0}$
$u[n]$	$\frac{1}{1-e^{-j\omega}}+\pi\sum_k \delta(\omega-2\pi k)$
$a^n u[n]$, $\|a\|<1$	$\frac{1}{1-ae^{-j\omega}}$
$(n+1)a^n u[n]$, $\|a\|<1$	$\frac{1}{(1-ae^{-j\omega})^2}$
$\frac{(n+r-1)!}{n!\,(r-1)!}a^n u[n]$, $\|a\|<1$	$\frac{1}{(1-ae^{-j\omega})^r}$
$a^{\|n\|}$, $\|a\|<1$	$\frac{1-a^2}{1-2a\cos\omega+a^2}$
1 on $\|n\|\le N_1$	$\frac{\sin(\omega(N_1+\frac{1}{2}))}{\sin(\omega/2)}$
$\frac{\sin(Wn)}{\pi n}$, $0<W<\pi$	1 on $\|\omega\|\le W$, 0 on $W<\|\omega\|\le\pi$
1	$2\pi\sum_k \delta(\omega-2\pi k)$
$e^{j\omega_0 n}$	$2\pi\sum_k \delta(\omega-\omega_0-2\pi k)$
$\cos(\omega_0 n)$	$\pi\sum_k[\delta(\omega-\omega_0-2\pi k)+\delta(\omega+\omega_0-2\pi k)]$
$\sin(\omega_0 n)$	$\frac{\pi}{j}\sum_k[\delta(\omega-\omega_0-2\pi k)-\delta(\omega+\omega_0-2\pi k)]$
$\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$	$2\pi\sum_k a_k\delta(\omega-\frac{2\pi k}{N})$
$\sum_k \delta[n-kN]$	$\frac{2\pi}{N}\sum_k \delta(\omega-\frac{2\pi k}{N})$

Written with the unnormalised sinc, $\operatorname{sinc}\theta=\sin\theta/\theta$, the low-pass sequence is $\frac{W}{\pi}\operatorname{sinc}(Wn)$.

What to carry into Chapter 7

- The pair, with analysis as the sum and synthesis as the integral over one period, carrying $\frac{1}{2\pi}$. It comes from the series: Na_k samples $X(e^{j\omega})$, and the samples fill in as N grows.
- The 2π-periodicity and its one-line proof, and that the same line fails in continuous time. High frequency means ω near $\pm\pi$.
- The DFT is N samples of the transform. Zero padding adds samples of the same curve; only a longer record narrows a peak, whose main lobe is $4\pi/L$ wide.
- Real means the phase is 0 or π, not that the phase is zero.
- A periodic sequence has impulses of weight $2\pi a_k$ at $\omega=2\pi k/N$ for every integer k.
- Multiplication in time is a **periodic** convolution over 2π, so anything reaching across a period boundary folds back and adds. A window turns each spectral line into a lobe; a softer window lowers the side lobes and widens the lobe.
- Parseval: $\sum_n |x[n]|^2=\frac{1}{2\pi}\int_{2\pi}|X(e^{j\omega})|^2\mathrm{d}\omega$, with $R=1\,\Omega$.
- A difference equation has $H(e^{j\omega})$ a ratio of polynomials in $e^{-j\omega}$; partial fractions give $h[n]$, and a repeated factor needs $(n+1)a^n u[n]$.

- The unnormalised sinc, $\operatorname{sinc}\theta = \sin\theta/\theta$, restated at every point of use.
- Discrete time has a difference, not a derivative; the derivative in this chapter is in frequency.

Many sequences are samples of a continuous-time signal, $x[n] = x(nT)$. The next chapter relates the two spectra and states when the samples are enough to rebuild $x(t)$. Sampling produces repeated spectral copies separated by the sampling frequency, so the periodic-overlap ideas of this chapter will be used again.

Exercises

6.1 Prove that $X(e^{j\omega})$ is 2π-periodic, and say precisely where the same argument fails for $X(j\omega)$.

Answer: Substituting $\omega + 2\pi$ into the analysis sum produces a factor $e^{-j2\pi n}$, which is 1 in every term because n is an integer. In continuous time the corresponding factor is $e^{-j2\pi t}$ with t real, which equals 1 only at integer t and cannot leave the integral.

6.2 Five ones, $x[n] = 1$ for $0 \le n \le 4$, and $N = 5$. How many of the five DFT values $X[k]$ are zero?

Answer: Four. $X(e^{j\omega}) = e^{-j2\omega}\sin(5\omega/2)/\sin(\omega/2)$, and at $\omega_k = 2\pi k/5$ the numerator is $\sin(\pi k) = 0$ for $k = 1, \ldots, 4$. Only $X[0] = 5$ is not zero.

6.3 The record $\cos(0.3\pi n) + \cos(0.4\pi n)$ is kept for $L = 10$ samples and padded to $N = 128$ (record A), or kept for $L = 30$ samples with no padding (record B). Whose curve $|X(e^{j\omega})|$ has two separate peaks?

Answer: Only B. Padding adds samples of the same curve, and at $L = 10$ the two lobes merge into one peak. At $L = 30$ the lobes are narrower and the two peaks part.

6.4 For $x[n] = a^n u[n]$ with $a = \frac{1}{8}$, give $|X|_{\max}$, $|X|_{\min}$ and $\max|\angle X|$ in closed form and as numbers.

Answer: $1/(1-a) = 8/7 = 1.1429$, $1/(1+a) = 8/9 = 0.8889$, and $\arcsin\frac{1}{8} = 0.1253$ rad.

6.5 Explain why the finite geometric sum behind the Dirichlet kernel needs $r \ne 1$ rather than $|r| < 1$, and give the value of the kernel at the excluded points.

Answer: A finite sum always has a value; only the closed form fails, where its denominator $1 - r$ vanishes. Here $|r| = 1$ at every ω, so $|r| < 1$ would exclude the sum entirely. At $r = 1$, that is ω a multiple of 2π, every term is 1 and the sum is $2N_1 + 1$.

6.6 Where is the first zero with $\omega > 0$ of the transform of the rectangular pulse with $N_1 = 3$? Does going from $N_1 = 2$ to $N_1 = 4$ halve the main lobe?

Answer: At $2\pi/7$, since the zeros are at $2\pi k/(2N_1 + 1)$. No: the first zero moves from $2\pi/5$ to $2\pi/9$, a factor $5/9$, the inverse ratio of the pulse lengths.

6.7 The Dirichlet kernel is real at every ω. Does it follow that $|X| = X$ and $\angle X = 0$?

Answer: No. Realness fixes the imaginary part and not the sign. The kernel is negative on part of every period, with least value -1.2500 for $N_1 = 2$, and there $|X| = -X$ and $\angle X = \pi$. The conclusion needs the extra hypothesis $X \ge 0$, which holds for $a^{|n|}$ but not here.

6.8 Invert the ideal low-pass spectrum with $W = \pi/2$ and give $x[0]$ and $x[2]$. State the sinc convention you use.

Answer: $x[n] = \sin(Wn)/(\pi n)$ with $x[0] = W/\pi = 0.5$, and $x[2] = \sin(\pi)/(2\pi) = 0$. In the unnormalised convention $\operatorname{sinc}\theta = \sin\theta/\theta$, the same sequence is $\frac{W}{\pi}\operatorname{sinc}(Wn)$.

6.9 Where do the impulses of $2\cos\left(\frac{5\pi}{3}n\right) + \cos\left(\frac{7\pi}{4}n\right)$ lie inside $-\pi < \omega \le \pi$, and what is the fundamental period of the sequence?
Answer: Weight 2π at $\pm\pi/3$ and weight π at $\pm\pi/4$, after reducing $5\pi/3$ to $\pi/3$ and $7\pi/4$ to $\pi/4$. The two components have periods 6 and 8, so the sum has period $\operatorname{lcm}(6,8) = 24$.

6.10 A turn of $3\pi/2$ per step on the unit circle gives the samples $\cos(3\pi n/2)$. Which turn between 0 and π gives the same samples?
Answer: $\pi/2$, since $2\pi - 3\pi/2 = \pi/2$ and $\cos((2\pi-\omega)n) = \cos(\omega n)$ at every integer n. The point turns the other way by $\pi/2$, and its real part is the same.

6.11 Two spectra, 1 on $|\omega| \le 3\pi/4$ and 1 on $|\omega| \le \pi/2$, belong to sequences that are multiplied together. Why is $Z(e^{j\pi}) = \frac{1}{4}$ and not $\frac{1}{8}$?
Answer: The multiplication property gives a periodic convolution over one period. The ordinary convolution of the two bands is a trapezoid reaching to $|\omega| = 5\pi/4$, which is wider than one period, so near $\omega = \pi$ a second copy arrives from the other side and the two contributions of $\frac{1}{8}$ add.

6.12 A cosine is kept for $L = 16$ samples. How wide is the main lobe of each line, and what happens to it for $L = 32$? What does a Hann window of the same length do to it?
Answer: $4\pi/L = \pi/4$ at $L = 16$ and $\pi/8$ at $L = 32$, half as wide. A Hann window doubles the main lobe to $8\pi/L$ and lowers the highest side lobe from about -13 dB to about -31 dB.

6.13 Derive the pair for $(n+1)a^n u[n]$ and say what fixes the exponent and the sign.
Answer: Differentiate $\sum_{n\ge0} a^n e^{-j\omega n} = 1/(1-ae^{-j\omega})$ with respect to a, shift the index and cancel one factor $e^{-j\omega}$, giving $1/(1-ae^{-j\omega})^2$. One differentiation of a first power produces the exponent 2, and the minus sign is inherited from the pair being differentiated.

6.14 Find the impulse response of $y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]$ and check it against the equation.
Answer: $h[n] = 4(\frac{1}{2})^n u[n] - 2(\frac{1}{4})^n u[n]$, so $h[0] = 2$, $h[1] = 1.5$, $h[2] = 0.875$. Substituting: $h[1] - \frac{3}{4}h[0] = 0$ and $h[2] - \frac{3}{4}h[1] + \frac{1}{8}h[0] = 0$.

6.15 An echo has $\alpha = 0.8$. Give the largest and smallest values of $|1 + \alpha e^{-j\omega D}|$, and say whether the recursion $w[n] = y[n] - \alpha w[n-D]$ is stable.
Answer: 1.8 where $e^{-j\omega D} = 1$ and 0.2 where $e^{-j\omega D} = -1$. The recursion is stable, because its impulse response $(-0.8)^k$ at $n = kD$ decays; its response lies between $1/1.8 = 0.5556$ and $1/0.2 = 5$.

6.16 Which properties of the continuous-time transform carry over unchanged, and which change form?
Answer: Linearity, the time shift, conjugate symmetry, Parseval and the convolution property carry over unchanged. The synthesis equation changes its range to one period; the multiplication property becomes a periodic convolution; the time-domain operator is a difference and not a derivative; and the transform pairs themselves differ, since the discrete-time ones are 2π-periodic and carry their own convergence conditions.

CHAPTER

7

Sampling and aliasing

Sampling keeps the value of a signal every T seconds and discards the values between those instants. This chapter gives the condition under which the samples still determine the signal, builds the filter that rebuilds it, and shows how the signal changes when the condition fails. It then processes the samples as numbers, and changes the rate of a sequence that is already sampled.

THE CHAPTER IN TWO LINES

$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\big(j(\omega - k\omega_s)\big)$$

$$\omega_s > 2\omega_M \implies x(t) \text{ can be recovered from } x(nT)$$

The first line says that sampling **replicates** the spectrum, at every rate. The second says that the copies stay apart when the rate is high enough. Copies that reach each other **overlap**, and only that overlap is **aliasing**. Keeping the two words apart is most of this chapter.

Section 7.1 models the sampler and derives the spectrum of its output. Section 7.2 defines aliasing and states the sampling theorem. Section 7.3 rebuilds the signal from its samples, first exactly and then with the holds a converter uses. Section 7.4 applies the alias rule to tones, to a chirp, to a camera and to an image. Section 7.5 processes the samples with a discrete-time system. Section 7.6 samples a sequence and changes its rate.

7.1 The sampler and the sampled spectrum

Impulse-train sampling

Sampling is modelled as a multiplication. The sampling function is an impulse train of period T, and the sampled signal is the product of the signal with it. The model keeps the sampled signal a function of continuous time, so the Fourier transform of Chapter 5 applies to it.

IMPULSE-TRAIN SAMPLING

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT), \qquad x_p(t) = x(t)\, p(t)$$

$$\begin{aligned} x_p(t) &= x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) \\ &= \sum_{n=-\infty}^{\infty} x(t)\, \delta(t - nT) \\ &= \sum_{n=-\infty}^{\infty} x(nT)\, \delta(t - nT) \end{aligned}$$

The first step substitutes $p(t)$ into the product. The second step moves $x(t)$ inside the sum, because multiplication distributes over addition. The third step uses the sifting property $x(t)\delta(t - nT) = x(nT)\delta(t - nT)$: an impulse is zero everywhere except at its own instant, so only the value of x at $t = nT$ survives.

HOW AN IMPULSE IS DRAWN

An impulse is drawn as an arrow whose **height is its weight**. In a picture of $x_p(t)$ the arrow at $t = nT$ therefore reaches $x(nT)$, and the arrowheads trace the signal itself. The arrows are weights, not values of a function. A sample equal to zero gives no arrow at all.

This chapter uses one running signal, $x(t) = \big(\sin(\pi t)/(\pi t)\big)^2$. Its transform is a triangle of peak 1 that reaches zero at $|\omega| = 2\pi$ rad/s (Example 7.5 derives a spectrum of this shape). So the highest angular frequency it carries is $\omega_M = 2\pi$ rad/s.

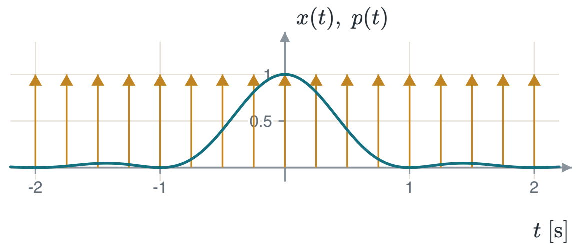

FIGURE 7.1 The running signal $x(t) = \big(\sin(\pi t)/(\pi t)\big)^2$ and the impulse train $p(t)$ with $T = 0.25$ s, each impulse of weight 1.

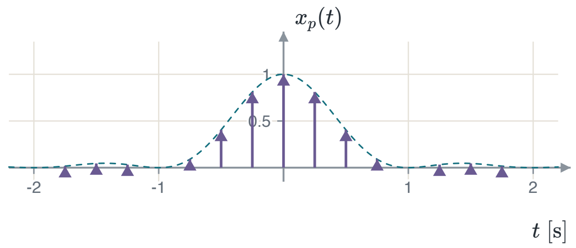

FIGURE 7.2 The product $x_p(t) = x(t)\, p(t)$. Each impulse now has weight $x(nT)$, so the arrowheads trace $x(t)$, dashed.

Samples as a sequence

The sampler keeps only the numbers $x(nT)$. Written as a sequence, they are

$$x_p[n] = x(nT), \qquad n = 0, \pm 1, \pm 2, \ldots$$

The rest of the chapter asks when these numbers are enough to rebuild $x(t)$. The period T spreads the arrows in time, but the sequence keeps exactly one number for each arrow. For the running signal, $x(t) = 0$ first at $t = 1$ s, because $\sin(\pi \cdot 1) = 0$. With $T = 0.2$ s the instant $nT = 1$ s is $n = 1/0.2 = 5$, so $x_p[5]$ is the first zero sample after $n = 0$.

The rate, in radians per second and in hertz

A sampling rate can be stated as an angular frequency or in hertz. Both forms are defined here so that their units stay distinct.

SAMPLING RATE, BOTH READINGS

$$\omega_s = \frac{2\pi}{T}\left[\frac{\text{rad}}{\text{s}}\right], \qquad f_s = \frac{1}{T}\,[\text{Hz}], \qquad \omega_s = 2\pi f_s$$

ω_s is the sampling **angular** frequency. It is the angle that $\cos(\omega_s t)$ turns through per second, and it turns through 2π rad in each period T. f_s is the sampling frequency in hertz, the number of samples taken per second. Every spectrum in Sections 7.1 to 7.4 is drawn against ω in rad/s, so ω_s is the working symbol; the hertz reading appears where a physical rate is quoted.

⊗ THE FACTOR OF 2π

Calling $2\pi/T$ the sampling frequency and then reading it in hertz is a costly slip. With $T = 0.25$ ms $= 2.5 \times 10^{-4}$ s the two readings are $\omega_s = 2\pi/(2.5 \times 10^{-4}) = 8000\pi$ rad/s and $f_s = 1/(2.5 \times 10^{-4}) = 4000$ Hz. They differ by 2π, and an answer that mixes them is wrong by that factor everywhere it is used. The same applies to the signal: a bandwidth of ω_M rad/s is $f_M = \omega_M/2\pi$ hertz.

One multiplication tests any pair of values: $\omega_s T = 2\pi$ always, and $f_s T = 1$ always. For example, a sensor that takes 200 samples per second has $f_s = 200$ Hz, so $\omega_s = 2\pi f_s = 2\pi \times 200 = 400\pi$ rad/s; the check is $400\pi \times \frac{1}{200} = 2\pi$.

The transform of the impulse train

Multiplication in time is convolution in frequency. So the spectrum of $x_p(t)$ needs the transform of the impulse train $p(t)$. Four steps give the result. Step 1 writes the product as a convolution. Step 2 finds $P(j\omega)$ from the Fourier series of $p(t)$. Steps 3 and 4 substitute P and evaluate the convolution with the sifting property.

Step 1 is the multiplication property of Chapter 5. A product in time is a convolution in frequency, divided by 2π; the $1/2\pi$ is the same factor that appears in the inverse transform.

$$X_p(j\omega) = \frac{1}{2\pi}\big[X(j\omega) * P(j\omega)\big] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)\, P\big(j(\omega-\theta)\big)\, \mathrm{d}\theta.$$

STEP 2A: THE FOURIER SERIES OF THE IMPULSE TRAIN

$$p(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_s t}, \qquad \omega_s = \frac{2\pi}{T}$$

$$\begin{aligned} a_k &= \frac{1}{T}\int_{-T/2}^{T/2} p(t)\, e^{-jk\omega_s t}\,\mathrm{d}t \\ &= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{-jk\omega_s t}\,\mathrm{d}t \\ &= \frac{1}{T}\, e^{-jk\omega_s \cdot 0} = \frac{1}{T} \end{aligned}$$

$p(t)$ is periodic with period T, so it has a Fourier series with fundamental frequency $\omega_s = 2\pi/T$. The coefficient integral runs over one period, $-T/2 < t < T/2$. Only the impulse at $n = 0$ lies inside that window, so $p(t)$ equals $\delta(t)$ there. The sifting property then evaluates the exponential at $t = 0$, where it equals 1. Every coefficient is the same number, $1/T$.

STEP 2B: THE TRANSFORM OF THE IMPULSE TRAIN

$$e^{jk\omega_s t} \longleftrightarrow 2\pi\,\delta(\omega - k\omega_s)$$

$$P(j\omega) = \sum_{k=-\infty}^{\infty} \frac{1}{T}\cdot 2\pi\,\delta(\omega - k\omega_s) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_s)$$

The first line is the transform pair of a complex exponential. The transform is linear, so each term of the series transforms separately and the constant $1/T$ passes through. An impulse train of spacing T in time is therefore an impulse train of spacing ω_s in frequency, with weight $2\pi/T$ on every impulse. For example, $T = 0.1$ s puts the impulses of $P(j\omega)$ $\omega_s = 2\pi/0.1 = 20\pi$ rad/s apart.

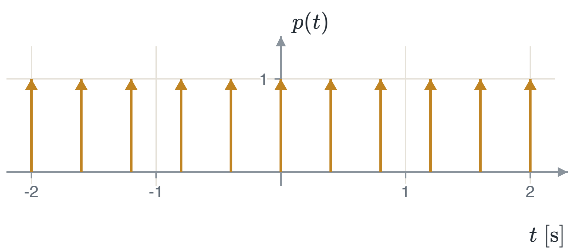

FIGURE 7.3 The impulse train $p(t)$ with $T = 0.4$ s: weight 1 every 0.4 s.

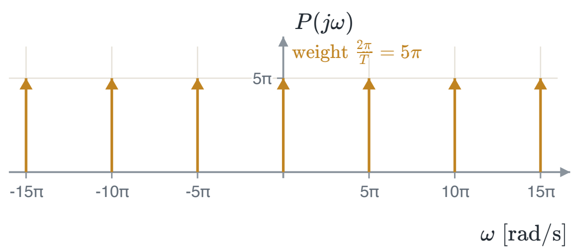

FIGURE 7.4 Its transform $P(j\omega)$: an impulse every $\omega_s = 5\pi$ rad/s, each drawn with height equal to its weight $2\pi/T = 5\pi$.

The spectrum of a sampled signal

Step 3 substitutes P from Step 2b into the convolution of Step 1. Step 4 evaluates the integral with the sifting property.

$$\begin{aligned} X_p(j\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)\,\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega-\theta-k\omega_s)\,\mathrm{d}\theta \\ &= \frac{1}{T}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} X(j\theta)\,\delta\big(\theta-(\omega-k\omega_s)\big)\,\mathrm{d}\theta \\ &= \frac{1}{T}\sum_{k=-\infty}^{\infty} X\big(j(\omega-k\omega_s)\big). \end{aligned}$$

In the second line the constants $1/2\pi$ and $2\pi/T$ multiply to $1/T$, and the sum is taken outside the integral. The impulse is even, so $\delta(\omega-\theta-k\omega_s)=\delta\big(\theta-(\omega-k\omega_s)\big)$; writing it this way shows where it sits on the θ axis. In the third line the sifting property picks out X at $\theta=\omega-k\omega_s$.

KEY RESULT: THE SPECTRUM OF THE SAMPLED SIGNAL

$$X_p(j\omega)=\frac{1}{T}\sum_{k=-\infty}^{\infty} X\big(j(\omega-k\omega_s)\big)$$

Each term is one copy of $X(j\omega)$, moved to $k\omega_s$ and scaled by $1/T$, not by 1. The reconstruction filter of Section 7.3 carries gain T, the inverse, so the two belong together. For example, if $X(j0)=2$, $T=0.05$ s and the copies do not overlap, only the $k=0$ copy reaches $\omega=0$, and $X_p(j0)=X(j0)/T=2/0.05=40$.

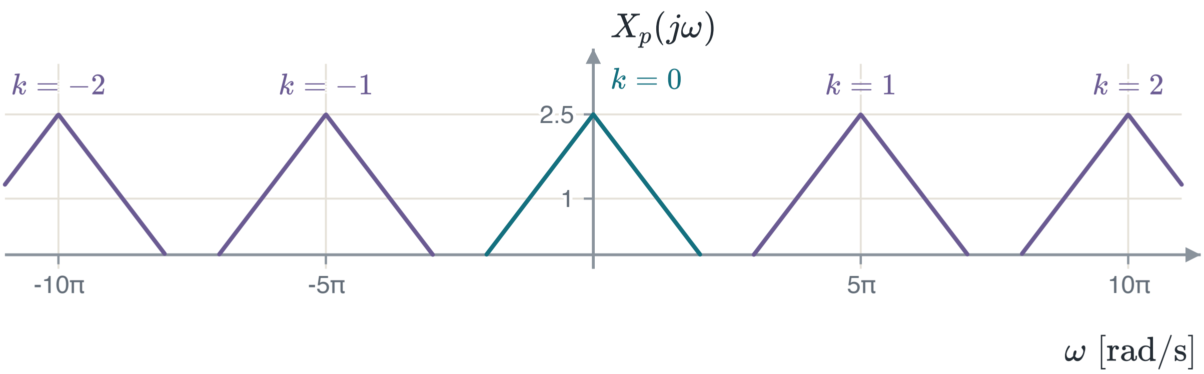

FIGURE 7.5 The sampled spectrum of the running signal at $T=0.4$ s, so $\omega_s=5\pi$ rad/s. Every copy has height $1/T=2.5$, and one sits at each multiple of ω_s.

Replication at every rate

The sum for $X_p(j\omega)$ runs over every integer k for every value of T. There is no rate at which the copies fail to appear, and no rate at which they disappear.

ⓘ NAMING THE PIECES

The term $k=0$ is $X(j\omega)/T$, the spectrum of the signal itself, where it always sat. It is the **baseband**. The terms $k=\pm1,\pm2,\ldots$ are the **copies**, centred at $\pm\omega_s,\pm2\omega_s,\ldots$

⊗ THE CENTRE IS NOT THE FIRST COPY

The baseband is $k=0$, not a copy. Calling it the first copy puts every later count off by one.

Only two things change with the rate. The copies sit $\omega_s = 2\pi/T$ apart, so a slower sampler (larger T) brings them closer. Each copy is scaled by $1/T$, so a slower sampler also makes every copy shorter. Halving T therefore doubles both the spacing $2\pi/T$ and the height $1/T$.

Sampling rate and guard band

The baseband occupies $|\omega| \le \omega_M$. The copy at $k = 1$ is $X\big(j(\omega - \omega_s)\big)/T$, which is non-zero where $|\omega - \omega_s| \le \omega_M$, that is on $\omega_s - \omega_M \le \omega \le \omega_s + \omega_M$. So it begins at $\omega_s - \omega_M$. The gap between the two decides everything.

GUARD BAND

$$\underbrace{(\omega_s - \omega_M)}_{\text{copy } k=1 \text{ starts}} - \underbrace{\omega_M}_{\text{baseband ends}} = \omega_s - 2\omega_M$$

The gap can be positive, zero or negative. For example, with $\omega_M = 3\pi$ rad/s and $\omega_s = 8\pi$ rad/s it is $8\pi - 2 \cdot 3\pi = 2\pi$ rad/s.

- $\omega_s > 2\omega_M$, **oversampling**: the gap is positive and the copies stand apart.
- $\omega_s = 2\omega_M$, **the Nyquist rate**: the gap is zero and the copies touch at $\pm\omega_M$.
- $\omega_s < 2\omega_M$, **undersampling**: the gap is negative, and neighbouring copies overlap and add.

⊗ TOUCHING IS ALREADY TOO LATE

At $\omega_s = 2\omega_M$ a filter edge placed at ω_M either drops a component that sits exactly there or takes in the edge of the next copy. So the rate must satisfy $\omega_s > 2\omega_M$, strictly. Example 7.1 shows a signal that loses a whole component at the boundary.

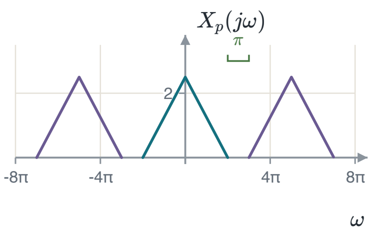

FIGURE 7.6 Oversampling, $\omega_s = 5\pi$ rad/s: a guard band of π rad/s.

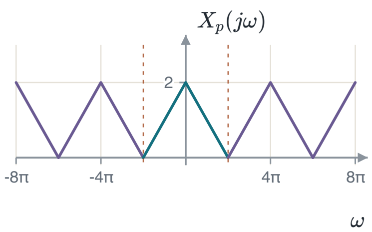

FIGURE 7.7 The Nyquist rate, $\omega_s = 4\pi = 2\omega_M$: the guard band is zero.

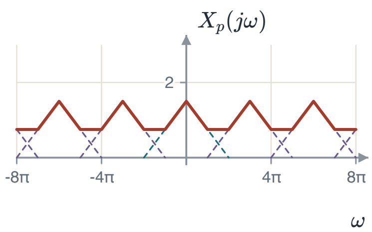

FIGURE 7.8 Undersampling, $\omega_s = 3\pi$ rad/s: the copies add. The solid red curve is their sum.

7.2 Aliasing and the sampling theorem

Aliasing

Aliasing is the overlap of neighbouring copies in $X_p(j\omega)$. The baseband ends at ω_M and the copy $k = 1$ starts at $\omega_s - \omega_M$. They overlap exactly when the copy starts before the baseband ends:

$$\omega_s - \omega_M < \omega_M \quad \Longleftrightarrow \quad \omega_s < 2\omega_M.$$

The overlap then runs from $\omega_s - \omega_M$ to ω_M, so its width is $\omega_M - (\omega_s - \omega_M) = 2\omega_M - \omega_s$. For the running signal, $\omega_M = 2\pi$ rad/s, sampled at $\omega_s = 3.5\pi$ rad/s, the overlap runs from 1.5π to 2π and is 0.5π rad/s wide.

> ⊗ REPLICATION IS NOT ALIASING
>
> It is sometimes said that below the Nyquist rate the spectrum is no longer replicated. That is false. The formula for $X_p(j\omega)$ carries no condition, so the copies are there at every rate; below the Nyquist rate they **overlap**. Replication is what sampling does. Aliasing is the overlap of the copies that sampling made.

Why aliasing cannot be undone

Where two copies meet, the sampler stores one number that is the sum of two contributions. Near the baseband, at $0 < \omega < \omega_M$, the two are the baseband and the copy $k = 1$:

$$X_p(j\omega) = \frac{1}{T}\Big[\underbrace{X(j\omega)}_{\text{wanted}} + \underbrace{X\big(j(\omega - \omega_s)\big)}_{\text{intruder}}\Big].$$

A filter multiplies $X_p(j\omega)$ frequency by frequency. It can keep or remove whole intervals of ω, but it cannot split one stored value into the two numbers that were added to make it. $X_p(j\omega)$ still equals $X(j\omega)/T$ alone only where no copy reaches, that is on $|\omega| < \omega_s - \omega_M$. At $\omega_s = 3.4\pi$ rad/s the running signal keeps this form only on $|\omega| < 3.4\pi - 2\pi = 1.4\pi$.

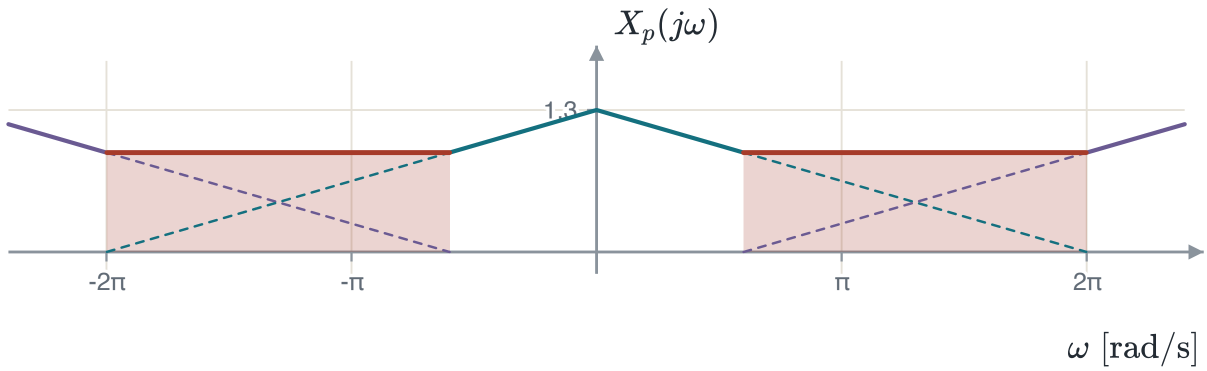

FIGURE 7.9 The running signal sampled at $\omega_s = 2.6\pi$ rad/s, near the baseband. Inside the red intervals the baseband, cyan, and a copy, violet, both contribute, and the sampler keeps only their sum, red.

> ✓ THE ONLY REPAIR IS PREVENTION
>
> Raise ω_s above $2\omega_M$, or remove the high frequencies of $x(t)$ **before** the sampler. The second is the anti-aliasing filter of Section 7.4.

The sampling theorem

KEY RESULT: THE SAMPLING THEOREM

$$X(j\omega) = 0 \ \text{for} \ |\omega| > \omega_M \quad \text{and} \quad \omega_s > 2\omega_M$$

$$\implies \quad x(t) \text{ is determined uniquely by } x(nT),\ n = 0, \pm 1, \pm 2, \ldots$$

The signal is recovered by passing $x_p(t)$ through an ideal low-pass filter of gain T and cutoff ω_c with $\omega_M < \omega_c < \omega_s - \omega_M$. The quantity $2\omega_M$ is the **Nyquist rate** of the signal, and ω_s must be strictly above it. The hypothesis matters: without a band limit there is no ω_M, and the theorem says nothing.

The cutoff interval comes from the guard band. The filter must pass the whole baseband, so $\omega_c > \omega_M$. It must stop the lowest copy, which starts at $\omega_s - \omega_M$, so $\omega_c < \omega_s - \omega_M$. For example, with $\omega_M = 2\pi$ and $\omega_s = 7\pi$ rad/s the interval is $2\pi < \omega_c < 7\pi - 2\pi = 5\pi$, so $\omega_c = 4\pi$ works, while 1.5π cuts into the baseband and 5.5π lets a copy through.

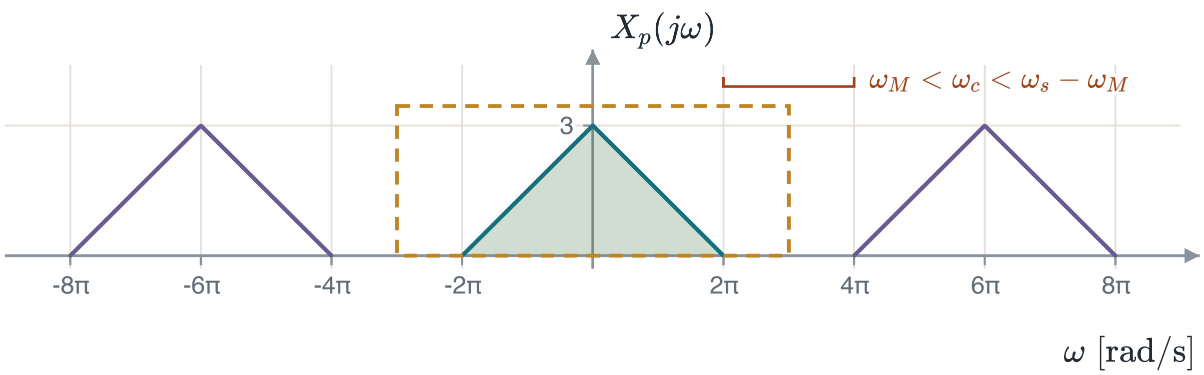

FIGURE 7.10 The running signal, $\omega_M = 2\pi$ rad/s, sampled at $\omega_s = 6\pi$ rad/s. The dashed amber box is a low-pass filter with $\omega_c = 3\pi$, the green area is what it passes, and the coral bracket is the interval in which ω_c may stand.

Why the inequality is strict

Put $\omega_s = 2\omega_M$ into the cutoff condition:

$$\omega_M < \omega_c < \underbrace{2\omega_M - \omega_M}_{\omega_s - \omega_M} = \omega_M.$$

No number ω_c satisfies $\omega_M < \omega_c < \omega_M$, so there is no admissible filter at the Nyquist rate itself. A statement with $\omega_s \geq 2\omega_M$ would promise a filter that its own cutoff condition rules out.

⚠ A REAL FILTER NEEDS A GAP

A real filter cannot jump from pass to stop; it needs room between the band edge and the first copy. So the rate is chosen with a guard band above $2\omega_M$. If the filter needs a gap of at least π rad/s and $\omega_M = 2\pi$ rad/s, then $\omega_s - 2\omega_M \geq \pi$ gives $\omega_s \geq 4\pi + \pi = 5\pi$ rad/s.

EXAMPLE 7.1 — THE NYQUIST-RATE BOUNDARY

GIVEN $x(t) = 1 + \cos(2000\pi t) + \sin(4000\pi t)$, sampled at exactly $\omega_s = 2\omega_M = 8000\pi$ rad/s.

FIND What is left of the term $\sin(4000\pi t)$ after sampling, and what a cosine at the same frequency would give.

METHOD Use the sampled-spectrum sum, because a copy can land on a baseband line and cancel it. Locate every contribution to X_p at $\omega = +4000\pi$, then confirm the result from the sample values in time.

SOLUTION The highest frequency present is 4000π rad/s, so $\omega_M = 4000\pi$ rad/s and

$$T = \frac{2\pi}{\omega_s} = \frac{2\pi}{8000\pi} = \frac{1}{4000}\ \mathrm{s} = 0.25\ \mathrm{ms}, \qquad \frac{1}{T} = 4000.$$

The sine transforms to $\frac{\pi}{j}\big[\delta(\omega - 4000\pi) - \delta(\omega + 4000\pi)\big]$; Example 7.3 derives this pair. Write the sampled-spectrum sum for this term and keep the $k = 0$ and $k = 1$ terms:

$$\begin{aligned} k = 0: &\quad \frac{1}{T}\cdot\frac{\pi}{j}\big[\delta(\omega - 4000\pi) - \delta(\omega + 4000\pi)\big] \\ k = 1: &\quad \frac{1}{T}\cdot\frac{\pi}{j}\big[\delta(\omega - 8000\pi - 4000\pi) - \delta(\omega - 8000\pi + 4000\pi)\big] \\ &= \frac{1}{T}\cdot\frac{\pi}{j}\big[\delta(\omega - 12000\pi) - \delta(\omega - 4000\pi)\big]. \end{aligned}$$

The $k = 1$ term replaces ω by $\omega - \omega_s$, which moves every impulse up by 8000π. The impulse that sat at -4000π now sits at $-4000\pi + 8000\pi = +4000\pi$, and it keeps its negative sign. At $\omega = +4000\pi$ the two weights add:

$$\frac{1}{T}\cdot\frac{\pi}{j} - \frac{1}{T}\cdot\frac{\pi}{j} = \frac{4000\pi}{j} - \frac{4000\pi}{j} = 0.$$

The same cancellation happens at -4000π, where the $k = -1$ copy brings the impulse from $+4000\pi$ down by 8000π. So the sine term is absent from the sampled signal. The cosine term $\cos(2000\pi t)$ is not affected: its line at -2000π moves to $-2000\pi + 8000\pi = 6000\pi$, which is outside the baseband.

CHECK The sample values of the sine are

$$\sin(4000\pi\, nT) = \sin\left(4000\pi \cdot \frac{n}{4000}\right) = \sin(\pi n) = 0 \quad \text{for every integer } n.$$

Every sample falls on a zero crossing, so the sampler never sees the term. A cosine at the same frequency behaves differently: $\cos(4000\pi n/4000) = \cos(\pi n) = (-1)^n$, so its samples are $+1$ and -1 in turn. At the boundary rate the result depends on the phase of the component, so the boundary rate cannot be trusted.

REPAIR Add a guard band. With $\omega_g = 1000\pi$ rad/s the rate becomes $\omega_s = 2\omega_M + \omega_g = 8000\pi + 1000\pi = 9000\pi$ rad/s, so

$$T = \frac{2\pi}{9000\pi} = \frac{1}{4500}\ \mathrm{s} \approx 222.2\ \mu\mathrm{s}.$$

The cutoff interval $\omega_M < \omega_c < \omega_s - \omega_M$ becomes $4000\pi < \omega_c < 9000\pi - 4000\pi = 5000\pi$ rad/s, which is no longer empty.

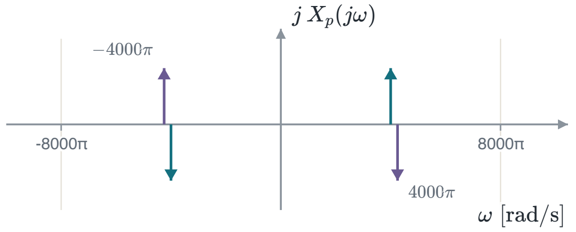

FIGURE 7.11 The sine term of $X_p(j\omega)$, times j. At $\pm 4000\pi$ the baseband line, cyan, meets a line of the opposite sign brought by the copy $k = \pm 1$, violet; they are drawn slightly apart and sum to zero.

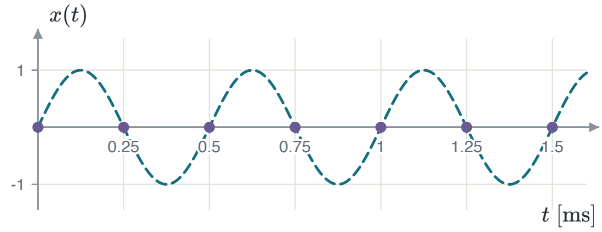

FIGURE 7.12 The band-edge tone $\sin(4000\pi t)$ with t in ms, and its samples at $T = 0.25$ ms. Every sample sits on a zero crossing.

EXAMPLE 7.2 — ONE SIGNAL AT THREE RATES

GIVEN The running signal $x(t) = \left(\sin(\pi t)/(\pi t)\right)^2$, whose transform is a triangle of peak 1 with $\omega_M = 2\pi$ rad/s. The periods are $T_1 = 0.40$ s, $T_2 = 0.50$ s and $T_3 = 2/3$ s.

FIND For each period: the rate, the guard band and the height of each copy. Which period lets $x(t)$ be recovered?

METHOD Use the three definitions of Section 7.1, because the question asks for the rate, the separation and the scale of the copies: $\omega_s = 2\pi/T$, guard band $\omega_s - 2\omega_M$ with $2\omega_M = 4\pi$, and copy height $1/T$ times the peak of 1.

SOLUTION With ω_s and the guard band in rad/s,

$$\begin{aligned}
T_1 = 0.40\ \text{s}: &\quad \omega_s = \frac{2\pi}{0.40} = 5\pi, \quad \text{guard} = 5\pi - 4\pi = +\pi, \quad \frac{1}{T_1} = \frac{1}{0.40} = 2.5\\
T_2 = 0.50\ \text{s}: &\quad \omega_s = \frac{2\pi}{0.50} = 4\pi, \quad \text{guard} = 4\pi - 4\pi = 0, \quad \frac{1}{T_2} = \frac{1}{0.50} = 2\\
T_3 = 2/3\ \text{s}: &\quad \omega_s = \frac{2\pi}{2/3} = 3\pi, \quad \text{guard} = 3\pi - 4\pi = -\pi, \quad \frac{1}{T_3} = \frac{3}{2} = 1.5
\end{aligned}$$

Dividing by $2/3$ is multiplying by $3/2$, which gives the last row. Only T_1 is safe. T_2 sits on the boundary, and at T_3 the copies overlap by π rad/s on each side.

CHECK $\omega_s T = 2\pi$ in each row: $5\pi \times 0.4 = 2\pi$, $4\pi \times 0.5 = 2\pi$, $3\pi \times \frac{2}{3} = 2\pi$. Also, from T_1 to T_3 the spacing falls from 5π to 3π and the height from 2.5 to 1.5: height and spacing move together, since both are proportional to $1/T$. A sketch with the same height at every rate hides half of what sampling does.

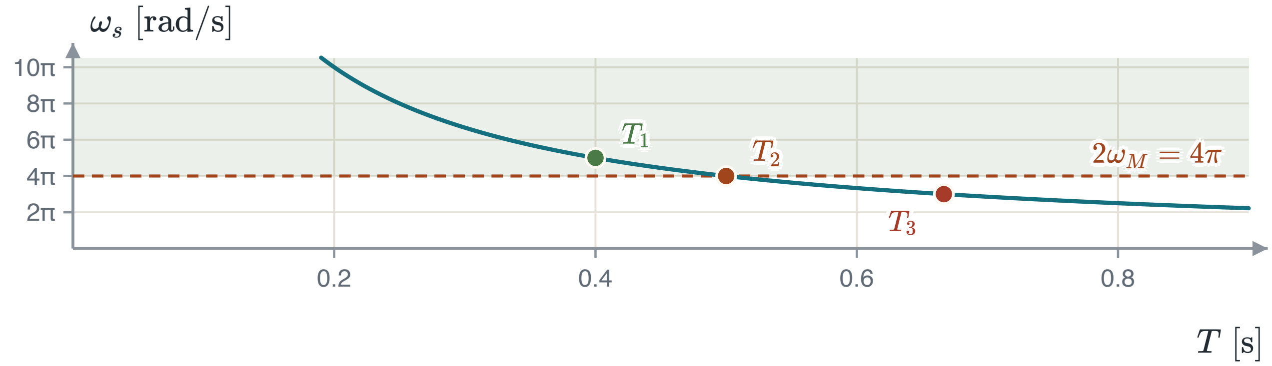

FIGURE 7.13 The rate $\omega_s = 2\pi/T$ against the period T. A period is safe while its point lies above the coral line $2\omega_M = 4\pi$ rad/s, in the green band.

EXAMPLE 7.3 — A LINE SPECTRUM AND ITS NYQUIST RATE

GIVEN $x(t) = 1 + \cos(2000\pi t) + \sin(4000\pi t)$.

FIND $X(j\omega)$, the bandwidth ω_M, and the Nyquist rate in rad/s and in hertz.

METHOD Use linearity, because the signal is a sum of standard terms. Write each term as a sum of complex exponentials and transform term by term with $e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)$. The farthest line from the origin gives ω_M.

SOLUTION The pair for one complex exponential holds because the inverse transform of the right side is $\frac{1}{2\pi}\int 2\pi\delta(\omega - \omega_0)e^{j\omega t}\,\mathrm{d}\omega = e^{j\omega_0 t}$ by sifting. Transform term by term:

$$\begin{aligned}
1 = e^{j0t} &\;\longleftrightarrow\; 2\pi\delta(\omega)\\
\cos(2000\pi t) &= \tfrac{1}{2}e^{j2000\pi t} + \tfrac{1}{2}e^{-j2000\pi t}\\
&\longleftrightarrow\; \pi\delta(\omega - 2000\pi) + \pi\delta(\omega + 2000\pi)\\
\sin(4000\pi t) &= \tfrac{1}{2j}e^{j4000\pi t} - \tfrac{1}{2j}e^{-j4000\pi t}\\
&\longleftrightarrow\; \frac{\pi}{j}\delta(\omega - 4000\pi) - \frac{\pi}{j}\delta(\omega + 4000\pi)
\end{aligned}$$

Each impulse weight is 2π times the coefficient of its exponential: $2\pi \cdot \frac{1}{2} = \pi$ and $2\pi \cdot \frac{1}{2j} = \pi/j$. Adding the three lines gives

$$\begin{aligned}
X(j\omega) = 2\pi\delta(\omega) &+ \pi\big[\delta(\omega - 2000\pi) + \delta(\omega + 2000\pi)\big]\\
&+ \frac{\pi}{j}\big[\delta(\omega - 4000\pi) - \delta(\omega + 4000\pi)\big].
\end{aligned}$$

Five impulses, each written as a function of ω so that its position can be read. The farthest sit at $\pm 4000\pi$, so

$$\begin{aligned}
\omega_M &= 4000\pi \text{ rad/s}, & f_M &= \frac{\omega_M}{2\pi} = 2000 \text{ Hz},\\
2\omega_M &= 8000\pi \text{ rad/s}, & 2f_M &= 4000 \text{ Hz}.
\end{aligned}$$

Any working rate must be strictly above 8000π rad/s.

CHECK Invert the impulses with $\frac{1}{2\pi}\int X(j\omega)e^{j\omega t}\,\mathrm{d}\omega$. The first gives $\frac{2\pi}{2\pi} = 1$. The cosine pair gives $\frac{\pi}{2\pi}\left(e^{j2000\pi t} + e^{-j2000\pi t}\right) = \frac{1}{2}\cdot 2\cos(2000\pi t) = \cos(2000\pi t)$. The sine pair gives

$\frac{\pi}{2\pi j}\left(e^{j4000\pi t} - e^{-j4000\pi t}\right) = \frac{1}{2j} \cdot 2j\sin(4000\pi t) = \sin(4000\pi t)$. The sum is $x(t)$. The signal repeats every 1 ms, and its fastest term runs at 2000 Hz.

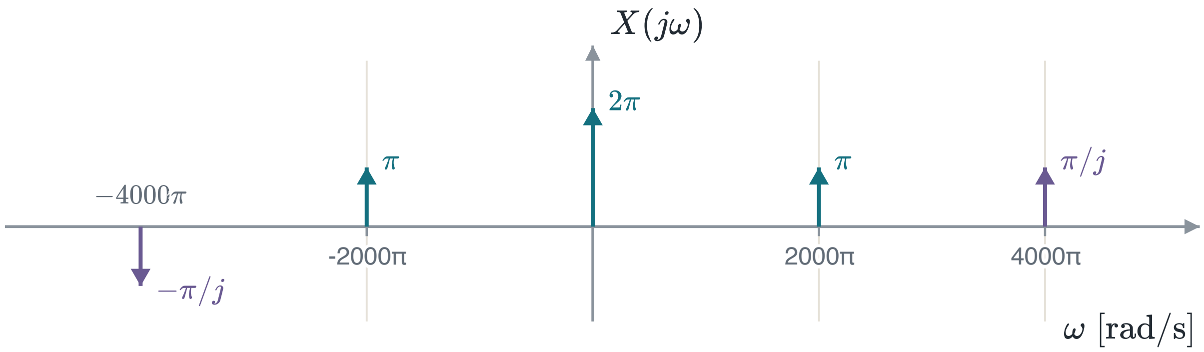

FIGURE 7.14 The line spectrum of Example 7.3. The cyan weights are real. The violet pair is imaginary, drawn up for $+\pi/j$ and down for $-\pi/j$.

EXAMPLE 7.4 – THE SAMPLING PERIOD AT THE NYQUIST RATE

GIVEN $$x(t) = \frac{\sin(4000\pi t)}{\pi t}.$$

FIND Its transform, the sampling period T at the Nyquist rate, and the height of each copy.

METHOD Invert a rectangle of height 1 to recognise the signal. The rectangle ends at ω_M, so $\omega_s = 2\omega_M$, and then $T = 2\pi/\omega_s$. Check the result three ways, because this is where a factor of 1000 is most easily lost.

SOLUTION Write $W = 4000\pi$ and let $R(j\omega)$ be 1 on $|\omega| \le W$ and 0 outside. Its inverse transform is

$$\begin{aligned}\frac{1}{2\pi}\int_{-W}^{W} 1 \cdot e^{j\omega t}\,\mathrm{d}\omega &= \frac{1}{2\pi}\left[\frac{e^{j\omega t}}{jt}\right]_{-W}^{W} = \frac{1}{2\pi}\cdot\frac{e^{jWt} - e^{-jWt}}{jt} \\ &= \frac{1}{2\pi}\cdot\frac{2j\sin(Wt)}{jt} = \frac{\sin(Wt)}{\pi t}.\end{aligned}$$

The bracket is the antiderivative of $e^{j\omega t}$ with respect to ω, and $e^{j\theta} - e^{-j\theta} = 2j\sin\theta$ removes the factor j from the denominator. So $X(j\omega) = R(j\omega)$, and $\omega_M = W = 4000\pi$ rad/s. The Nyquist rate is $\omega_s = 2\omega_M = 8000\pi$ rad/s, and

$$T = \frac{2\pi}{\omega_s} = \frac{2\pi}{8000\pi} = \frac{1}{4000}\ \mathrm{s} = 2.5 \times 10^{-4}\ \mathrm{s} = 0.25\ \mathrm{ms}.$$

In hertz the rate is $f_s = 1/T = 4000$ Hz, and each copy of the rectangle is scaled to height $1/T = 4000$. At this rate neighbouring copies touch at $\pm 4000\pi$.

CHECK Three independent tests. First, $\omega_s T = 8000\pi \times 2.5 \times 10^{-4} = 2\pi$; with 0.25 s it would be 2000π instead. Second, the copy scale is $1/T = 4000$; with 0.25 s it would be 4. Third, doubling the rate must halve the period: at 16000π rad/s, $T = 2\pi/(16000\pi) = 1/8000$ s $= 125$ μs, and 0.25 ms divided by 125 μs is exactly 2.

WARNING The answer is a quarter of a millisecond. Cancelling the π but not the thousand gives 0.25 s, and every later number is then wrong by a factor of 1000. A quarter of a second between samples would be four samples a second for a signal that carries 2000 Hz: the next sample would come $0.25 \times 2000 = 500$ cycles of the tone later.

EXAMPLE 7.5 — AREA AND PEAK OF A TRIANGULAR SPECTRUM

GIVEN $x(t) = \big(\sin(4000\pi t)/\pi t\big)^2$, the square of the signal of Example 7.4, whose transform $R(j\omega)$ is 1 on $|\omega| \le 4000\pi$.

FIND The peak of $X(j\omega)$, the bandwidth, the period at the Nyquist rate, and the height of one copy after sampling at that rate.

METHOD Use the multiplication property, because the signal is squared in time. The spectrum is then the rectangle convolved with itself, divided by 2π: a triangle. Keep its area and its peak apart.

SOLUTION Write $W = 4000\pi$ and $g(t) = \sin(Wt)/\pi t$, so that $x(t) = g(t)^2$ and $G(j\omega) = R(j\omega)$ by Example 7.4. Squaring in time is convolution in frequency divided by 2π:

$$X(j\omega) = \frac{1}{2\pi}\big[R * R\big](\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(j\theta)\, R\big(j(\omega-\theta)\big)\, \mathrm{d}\theta.$$

The integrand is 1 where both rectangles are 1, that is where $-W \le \theta \le W$ and $\omega - W \le \theta \le \omega + W$ at the same time, and 0 elsewhere. For $0 \le \omega \le 2W$ the overlap runs from $\omega - W$ to W and gives

$$\big[R * R\big](\omega) = \int_{\omega-W}^{W} 1\, \mathrm{d}\theta = W - (\omega - W) = 2W - \omega.$$

For negative ω the same argument gives $2W + \omega$, so $[R * R](\omega) = 2W - |\omega|$ on $|\omega| \le 2W$ and 0 outside: a triangle. Its value at the origin, A, is the area of the full rectangle, and the peak of X is A divided by 2π:

$$A = \big[R * R\big](0) = \int_{-W}^{W} (1)(1)\, \mathrm{d}\theta = 2W = 8000\pi,$$
$$X_{\max} = X(j0) = \frac{A}{2\pi} = \frac{8000\pi}{2\pi} = 4000.$$

The triangle reaches zero at $2W$, twice the half-width of the rectangle, so $\omega_M = 8000\pi$ rad/s. The Nyquist rate is $2\omega_M = 16000\pi$ rad/s, and

$$T = \frac{2\pi}{16000\pi} = \frac{1}{8000}\ \mathrm{s} = 1.25 \times 10^{-4}\ \mathrm{s} = 125\ \mu\mathrm{s}.$$

The copy height is $X_{\max}/T = 4000 \times 8000 = 3.2 \times 10^7$.

CHECK This bandwidth is twice the one in Example 7.4, so the period must be half of it: $125\ \mu$s against 0.25 ms. Also $\omega_s T = 16000\pi \times 1.25 \times 10^{-4} = 2\pi$.

WARNING A is an area and $X_{\max}$ is the peak of a spectrum. They differ by 2π, so substituting one where the other belongs inflates every later height by that factor. Every later formula uses $X_{\max}$, never A.

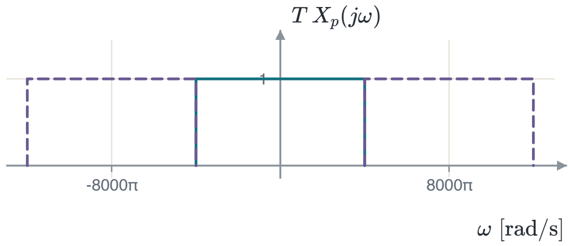

FIGURE 7.15 Example 7.4 at the Nyquist rate 8000π rad/s, drawn times T. The rectangle and its copies touch at $\pm 4000\pi$.

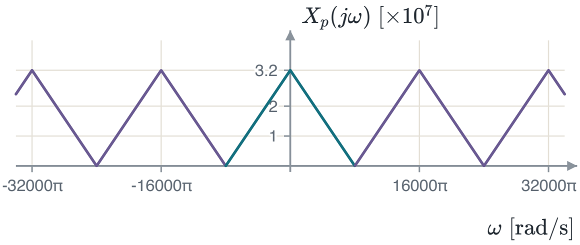

FIGURE 7.16 Example 7.5 at its Nyquist rate 16000π rad/s. Each copy stands 3.2×10^7 tall, and neighbours just touch.

Band-pass sampling

A **band-pass** signal has $X(j\omega) = 0$ outside $\omega_L < |\omega| < \omega_H$. Its width is $B = \omega_H - \omega_L$. The theorem asks for $\omega_s > 2\omega_H$, but that condition is sufficient, not necessary. The spectrum is empty below ω_L, and at a lower rate the copies can land in that empty space without meeting.

The positive band occupies $\omega_L < \omega < \omega_H$, and its copies occupy $\omega_L + k\omega_s < \omega < \omega_H + k\omega_s$. The negative band occupies $-\omega_H < \omega < -\omega_L$, and its copies occupy $-\omega_H + m\omega_s < \omega < -\omega_L + m\omega_s$. Every copy is a shift of one of these two by a multiple of ω_s, so the picture near one gap repeats in every gap. It is therefore enough to fit one copy of the negative band into the gap between the positive-band copy $k = -1$, which ends at $\omega_H - \omega_s$, and the positive band itself, which starts at ω_L. For the copy m this asks

$$\begin{aligned} \omega_H - \omega_s \le -\omega_H + m\omega_s &\iff (m+1)\,\omega_s \ge 2\omega_H, \\ -\omega_L + m\omega_s \le \omega_L &\iff m\,\omega_s \le 2\omega_L. \end{aligned}$$

Write $n = m + 1$. The two conditions together give one window of rates for each positive integer n:

> THE BAND-PASS SAMPLING WINDOWS
>
> $$\frac{2\omega_H}{n} \le \omega_s \le \frac{2\omega_L}{n-1}, \qquad n = 1, 2, \ldots, \left\lfloor \frac{\omega_H}{B} \right\rfloor$$
>
> For $n = 1$ the right side has $n - 1 = 0$ in its denominator and gives no upper limit: this is $\omega_s \ge 2\omega_H$. A window is not empty only when $2\omega_H/n \le 2\omega_L/(n-1)$. Multiply by $n(n-1)/2$: $(n-1)\omega_H \le n\omega_L$, so $n(\omega_H - \omega_L) \le \omega_H$, that is $n \le \omega_H/B$. At an end of a window the copies touch; as at the Nyquist rate, a component exactly at a band edge is then at risk. When the copies stay apart, a band-pass filter of gain T on $\omega_L < |\omega| < \omega_H$ returns $x(t)$.

EXAMPLE 7.6 – BAND-PASS SAMPLING

GIVEN A signal whose spectrum fills $8\pi < |\omega| < 10\pi$ rad/s.

FIND Every rate at which the copies do not overlap. Does 7π work? Does 9π?

METHOD Compute B and $\lfloor \omega_H/B \rfloor$, then list the windows $2\omega_H/n \le \omega_s \le 2\omega_L/(n-1)$. Check the lowest rate by moving each half of the band.

SOLUTION $\omega_L = 8\pi$, $\omega_H = 10\pi$, so $B = 2\pi$ and $\omega_H/B = 5$. The lower edge is $4B$. The windows are

$$\begin{aligned}
n = 1: &\quad \omega_s \ge 20\pi \\
n = 2: &\quad 10\pi \le \omega_s \le 16\pi \\
n = 3: &\quad \tfrac{20\pi}{3} \approx 6.67\pi \le \omega_s \le 8\pi \\
n = 4: &\quad 5\pi \le \omega_s \le \tfrac{16\pi}{3} \approx 5.33\pi \\
n = 5: &\quad 4\pi \le \omega_s \le 4\pi.
\end{aligned}$$

So 7π works, because it lies in the window $n = 3$. The rate 9π does not, because it lies between 8π and 10π, in no window. A faster rate is not always a safer one. The lowest rate is $\omega_s = 2B = 4\pi$ rad/s, five times lower than $2\omega_H = 20\pi$.

CHECK

At $\omega_s = 4\pi$ move each half of the band by a multiple of ω_s into $0 \le \omega \le 4\pi$:

$$\begin{aligned}
8\pi < \omega < 10\pi &\xrightarrow{-2\omega_s} 0 < \omega < 2\pi, \\
-10\pi < \omega < -8\pi &\xrightarrow{+3\omega_s} 2\pi < \omega < 4\pi.
\end{aligned}$$

The two halves take turns and fill the axis without overlapping. If the band moves up to $12\pi < |\omega| < 14\pi$, then $\omega_H/B = 7$ and the window $n = 7$ is $4\pi \le \omega_s \le 4\pi$: the lowest rate is again $2B = 4\pi$, because the lower edge is again a whole multiple of B.

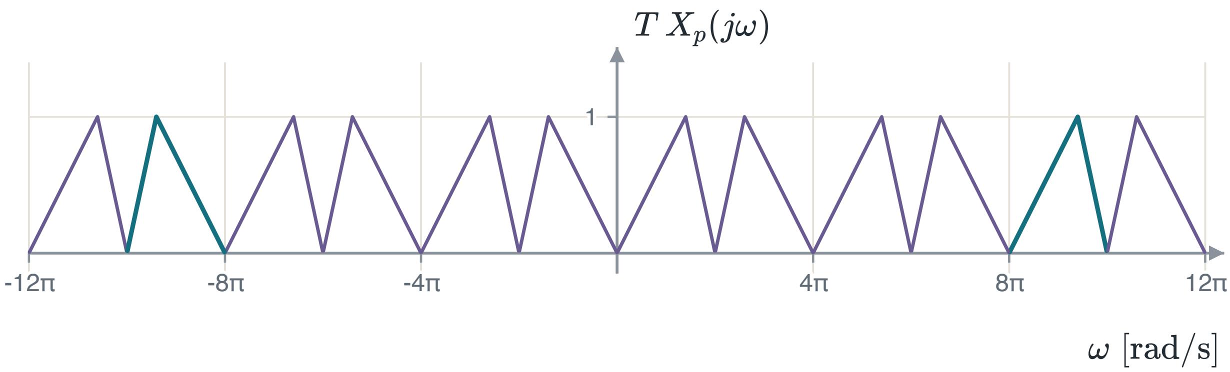

FIGURE 7.17 The band on $8\pi < |\omega| < 10\pi$ rad/s, cyan, and its copies at $\omega_s = 4\pi$ rad/s, violet, drawn times T. The band leans, so a copy of the positive half and a copy of the negative half can be told apart; they take turns along the axis.

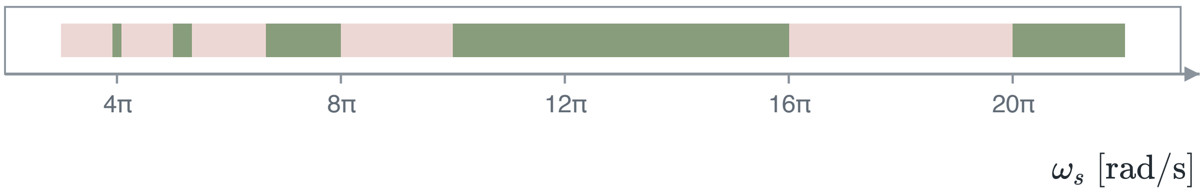

FIGURE 7.18 The rates from 3π to 22π rad/s for the band of Example 7.6: green where the copies fit, red where they overlap. The green windows are the point 4π, then 5π to 5.33π, 6.67π to 8π, 10π to 16π, and everything from 20π up.

7.3 Reconstruction

The ideal reconstruction filter

If the copies stand clear, recovering $x(t)$ means keeping the one at the origin and discarding the rest. A low-pass filter does exactly that.

IDEAL RECONSTRUCTION FILTER

$$H_r(j\omega) = \begin{cases} T, & |\omega| < \omega_c \\ 0, & |\omega| > \omega_c \end{cases} \qquad \omega_M < \omega_c < \omega_s - \omega_M$$

$$X_r(j\omega) = H_r(j\omega)\, X_p(j\omega) = T \cdot \tfrac{1}{T} X(j\omega) = X(j\omega) \implies x_r(t) = x(t)$$

Filtering multiplies the spectra. Inside $|\omega| < \omega_c$ only the $k = 0$ term of the sum for $X_p(j\omega)$ is non-zero, and that term is $X(j\omega)/T$; every other copy starts at $\omega_s - \omega_M$ or beyond, which is outside the passband. The gain is T because the baseband is $X(j\omega)/T$. The second line holds only when the copies do not overlap.

ⓧ COMMON ERROR: GAIN 1

A filter of gain 1 returns $X(j\omega)/T$, that is $x(t)/T$, not $x(t)$. Since T changes with the rate, the error changes with it.

For example, a signal with $\omega_M = 2\pi$ rad/s sampled at $\omega_s = 10\pi$ rad/s needs $2\pi < \omega_c < 10\pi - 2\pi = 8\pi$. So $\omega_c = 5\pi$ recovers $x(t)$, while $\omega_c = \pi$ cuts the baseband and $\omega_c = 9\pi$ lets the copies at $\pm 10\pi$ through.

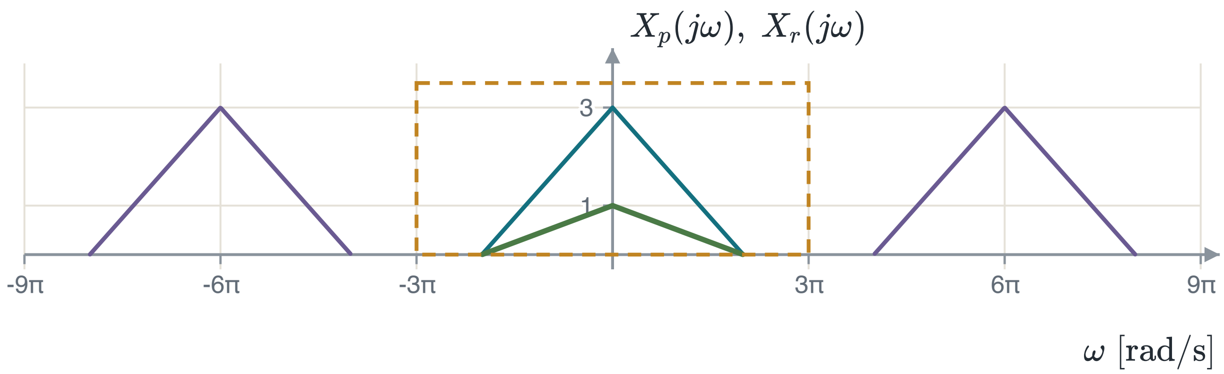

FIGURE 7.19 The running signal with $T = 1/3$ s, so $\omega_s = 6\pi$ rad/s and each copy has height $1/T = 3$. The dashed amber filter has gain T on $|\omega| < 3\pi$. It removes the copies at $\pm 6\pi$, violet, and scales the baseband, cyan, down to the green $X_r(j\omega) = X(j\omega)$ of height 1.

The chain in time

Two stages turn $x(t)$ into $x_r(t)$. The sampler turns $x(t)$ into $x_p(t) = \sum_n x(nT)\,\delta(t - nT)$, an impulse train of period T weighted by the samples. The filter $H_r(j\omega)$ turns $x_p(t)$ into $x_r(t)$. Between two samples, at $t = nT + T/2$, the impulse train is zero; only the filter output has a value there.

ⓧ NAME THE SIGNALS APART

The sampler output is $x_p(t)$. The name $x_r(t)$ belongs to the output of the reconstruction filter and to nothing before it. A diagram that labels the sampler output $x_r(t)$ has skipped the filter.

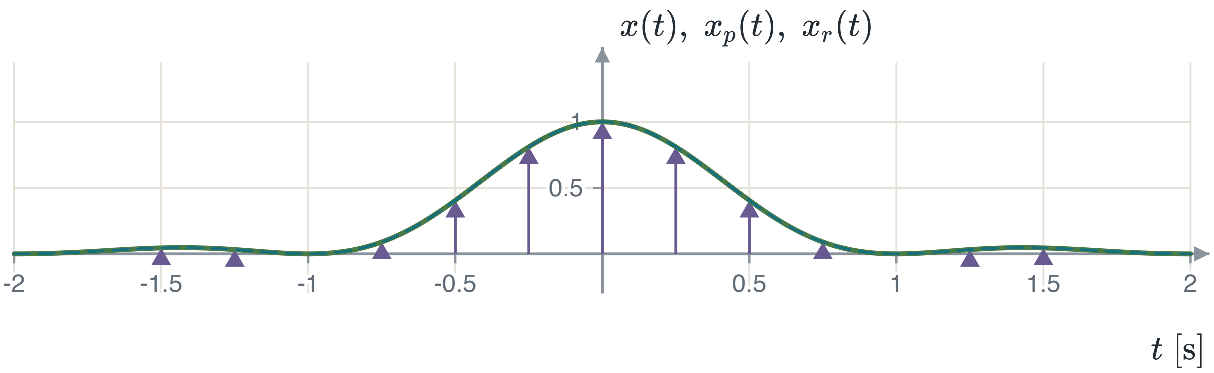

FIGURE 7.20 The running signal, dashed, the sampler output $x_p(t)$ with $T = 0.25$ s, violet, and the filter output $x_r(t)$ with $\omega_c = \pi/T = 4\pi$ rad/s, green. The green curve lies on $x(t)$.

Band-limited interpolation

Filtering is convolution with the impulse response $h_{LP}(t)$ of the filter, and x_p is a train of impulses. So the output is a sum of shifted copies of h_{LP}: each sample is replaced by a curve of its own height, and the curves are added. The first chain below finds h_{LP} as the inverse transform of H_r; the second forms the convolution.

THE KERNEL AND THE INTERPOLATION FORMULA

$$\begin{aligned} h_{LP}(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} H_r(j\omega)\, e^{j\omega t}\,\mathrm{d}\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} T\, e^{j\omega t}\,\mathrm{d}\omega \\ &= \frac{T}{2\pi}\left[\frac{e^{j\omega t}}{jt}\right]_{-\omega_c}^{\omega_c} = \frac{T}{2\pi}\cdot\frac{e^{j\omega_c t} - e^{-j\omega_c t}}{jt} \\ &= \frac{T}{2\pi}\cdot\frac{2j\sin(\omega_c t)}{jt} = \frac{T\sin(\omega_c t)}{\pi t} \end{aligned}$$

$$\begin{aligned} x_r(t) = x_p(t) * h_{LP}(t) &= \int_{-\infty}^{\infty}\Big[\sum_{n=-\infty}^{\infty} x(nT)\,\delta(\tau - nT)\Big] h_{LP}(t-\tau)\,\mathrm{d}\tau \\ &= \sum_{n=-\infty}^{\infty} x(nT)\int_{-\infty}^{\infty}\delta(\tau - nT)\, h_{LP}(t-\tau)\,\mathrm{d}\tau = \sum_{n=-\infty}^{\infty} x(nT)\, h_{LP}(t - nT) \end{aligned}$$

In the first chain the limits shrink to $\pm\omega_c$ because H_r is zero outside them, and T comes out as a constant. The bracket is the antiderivative of $e^{j\omega t}$ with respect to ω; then $e^{j\theta} - e^{-j\theta} = 2j\sin\theta$ removes the j, and $2/2\pi = 1/\pi$. At $t = 0$ the integral is $\frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} T\,\mathrm{d}\omega = T\omega_c/\pi$, which is also the limit of $T\sin(\omega_c t)/(\pi t)$. The second chain is the convolution integral with the impulse train substituted for x_p; the sum is taken outside the integral, and the sifting property evaluates h_{LP} at $\tau = nT$ for each n.

KEY RESULT: BAND-LIMITED INTERPOLATION

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)\, h_{LP}(t - nT) = \sum_{n=-\infty}^{\infty} x(nT)\,\frac{T\sin\big(\omega_c(t - nT)\big)}{\pi(t - nT)}$$

One kernel per sample, scaled by it. The sum is linear in the samples. If a fault doubles one sample, so that $x(0)$ becomes $2x(0)$, the output changes by $x(0)\,h_{LP}(t)$, and that change spreads over all t, not only $t = 0$.

The interpolation kernel

The cutoff can be anywhere in $\omega_M < \omega_c < \omega_s - \omega_M$. The middle of that interval is $\frac{1}{2}\big(\omega_M + \omega_s - \omega_M\big) = \omega_s/2 = \pi/T$, and it is the usual choice. With $\omega_c = \pi/T$ the kernel becomes

$$h_{LP}(t) = \frac{T\sin(\pi t/T)}{\pi t} = \frac{\sin(\pi t/T)}{\pi t/T} = \mathrm{sinc}\Big(\frac{\pi t}{T}\Big), \qquad \mathrm{sinc}(\theta) = \frac{\sin\theta}{\theta}.$$

The second equality divides numerator and denominator by T. Throughout this course $\mathrm{sinc}(\theta) = \sin\theta/\theta$ is **unnormalised**. At $t = 0$ the kernel is $T\omega_c/\pi = T(\pi/T)/\pi = 1$. At $t = mT$ with $m \neq 0$ it is $\sin(\pi m)/(\pi m) = 0$. So each sample sets the output at its own instant and adds nothing at the other sample instants: $x_r(mT) = x(mT)$. Another cutoff loses this property. With $\omega_c = 1.5\pi/T$, for example, $h_{LP}(0) = T \cdot 1.5\pi/(\pi T) = 1.5$.

> ⊗ KEEP THE π INSIDE THE SINE
>
> The kernel is $\mathrm{sinc}(\pi t/T) = \sin(\pi t/T)/(\pi t/T)$. A kernel written $\sin(t/T)/(\pi t/T)$ has lost a π: its zeros move to $t = m\pi T$, off the sample instants. With $T = 1$ the correct kernel at $t = 1$ is $\sin(\pi)/\pi = 0$, while the damaged one gives $\sin(1)/\pi = 0.267849$, so that sample leaks into its neighbour. At $t = 0$ the damaged kernel is not even 1: near $t = 0$, $\sin t \approx t$, so $\sin(t)/(\pi t) \to 1/\pi \approx 0.318$. Anything written $\mathrm{sinc}(t/T)$ belongs to the other, normalised convention and is a different function.

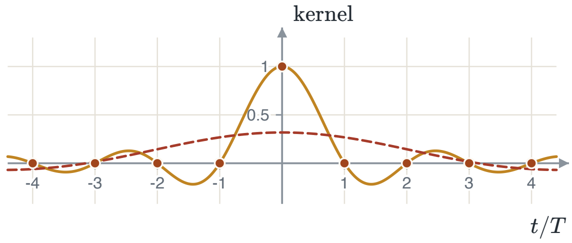

FIGURE 7.21 The kernel $\mathrm{sinc}(\pi t/T)$, amber, is 1 at its own instant and 0 at every other one. With the π dropped, dashed red, it misses the instants.

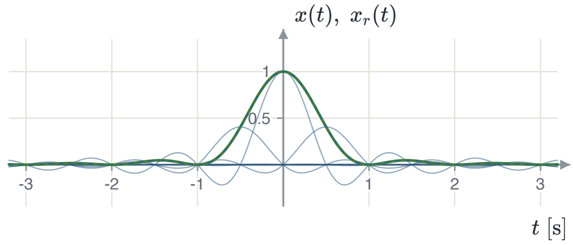

FIGURE 7.22 The running signal sampled with $T = 0.5$ s: one kernel per sample, grey, and their sum, green. It lands on the dashed original.

The zero-order hold

The ideal filter cannot be built: its impulse response starts before $t = 0$ and never ends. A converter holds each sample instead, until the next one arrives.

> ZERO-ORDER HOLD
>
> $$h_0(t) = \begin{cases} 1, & 0 \le t < T \\ 0, & \text{otherwise} \end{cases} \qquad x_0(t) = x(nT), \quad nT \le t < (n+1)T$$
>
> The hold replaces each impulse $x(nT)\,\delta(t - nT)$ by a rectangle of height $x(nT)$ and width T, so its output is a staircase.

The staircase is not $x(t)$. Over one tread, from nT to $(n+1)T$, a smooth signal moves away from the held value by about $|x'(t)|\,T$, because $x(nT + \tau) \approx x(nT) + x'(nT)\,\tau$ and τ reaches T. So the largest gap scales with T: halving T from 0.2 s to 0.1 s about halves it, but it never reaches zero at any finite rate.

THE FREQUENCY RESPONSE OF THE HOLD

$$
\begin{aligned}
H_0(j\omega) &= \int_{-\infty}^{\infty} h_0(t)\, e^{-j\omega t}\, \mathrm{d}t = \int_0^T e^{-j\omega t}\, \mathrm{d}t = \left[\frac{e^{-j\omega t}}{-j\omega}\right]_0^T = \frac{1-e^{-j\omega T}}{j\omega} \\
&= \frac{e^{-j\omega T/2}\left(e^{j\omega T/2} - e^{-j\omega T/2}\right)}{j\omega} = \frac{e^{-j\omega T/2}\cdot 2j\sin(\omega T/2)}{j\omega} \\
&= e^{-j\omega T/2}\,\frac{2\sin(\omega T/2)}{\omega} = T\, e^{-j\omega T/2}\,\frac{\sin(\omega T/2)}{\omega T/2}
\end{aligned}
$$

The limits shrink to 0 and T because h_0 is zero elsewhere. The bracket is the antiderivative of $e^{-j\omega t}$; evaluating it at T and at 0 gives $(e^{-j\omega T}-1)/(-j\omega)$, which is the fraction shown. The second line factors $e^{-j\omega T/2}$ out of $1-e^{-j\omega T}$, which leaves $e^{j\omega T/2}-e^{-j\omega T/2} = 2j\sin(\omega T/2)$; the j then cancels. The last form multiplies and divides by $T/2$ to show the unnormalised sinc: $H_0(j\omega) = T\,e^{-j\omega T/2}\,\mathrm{sinc}(\omega T/2)$ with $\mathrm{sinc}(\theta) = \sin\theta/\theta$. The factor $e^{-j\omega T/2}$ is a delay of half a sampling period.

Three readings of $|H_0(j\omega)| = |2\sin(\omega T/2)/\omega|$ compare it with the ideal filter of gain T and cutoff $\omega_s/2 = \pi/T$.

- At $\omega = 0$, $\sin\theta/\theta \to 1$, so $|H_0(j0)| = T$: the hold already has the gain the ideal filter needs.
- Inside the band it **sags**. At the edge $\omega = \pi/T$: $|H_0| = 2\sin(\pi/2)/(\pi/T) = 2T/\pi \approx 0.64\,T$.
- Outside the band it **leaks**: it is small but not zero, so parts of the copies remain. Its first zero for $\omega > 0$ is where $\omega T/2 = \pi$, that is $\omega = 2\pi/T = \omega_s$. With $T = 0.1$ s that is 20π rad/s.

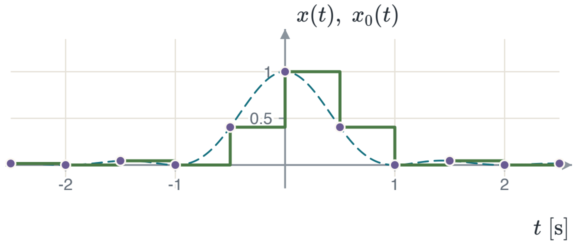

FIGURE 7.23 The zero-order hold with $T = 0.5$ s: each sample of the running signal is held for one period.

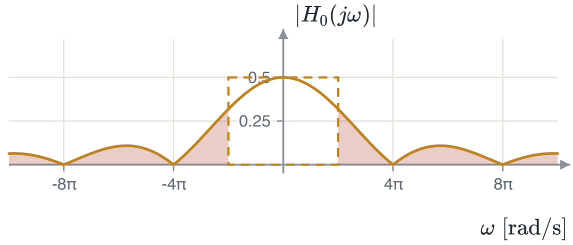

FIGURE 7.24 $|H_0(j\omega)|$ with $T = 0.5$ s against the ideal filter, dashed: flat at T up to $\omega_s/2 = 2\pi$ rad/s. The hold sags inside the band and leaks outside it, red.

EXAMPLE 7.7 — HEARING THE HOLD

GIVEN A tone $x(t) = \cos(2\pi f_0 t)$ with $f_0 = 300$ Hz, sampled at f_s samples per second and played through a zero-order hold.

FIND The level of the first copy of the tone, relative to the tone, at $f_s = 2000$ Hz and at $f_s = 8000$ Hz.

METHOD Sampling puts copies of the tone near $f_s \pm f_0$, $2f_s \pm f_0$, and so on; the nearest is at $f_s - f_0$. The hold multiplies each line by $|H_0|$, so compare $|H_0|$ at $f_s - f_0$ with $|H_0|$ at f_0. Write the response in hertz with $\omega = 2\pi f$ and $T = 1/f_s$.

SOLUTION Substitute $\omega = 2\pi f$ and $T = 1/f_s$ into $|H_0| = |2\sin(\omega T/2)/\omega|$:

$$|H_0(j2\pi f)| = \frac{2|\sin(\pi f/f_s)|}{2\pi f} = \frac{|\sin(\pi f/f_s)|}{\pi f}.$$

At the copy, $f = f_s - f_0$, the sine is $\sin\big(\pi - \pi f_0/f_s\big) = \sin(\pi f_0/f_s)$, by $\sin(\pi - \theta) = \sin\theta$. So the sines cancel in the ratio:

$$\frac{|H_0(j2\pi(f_s - f_0))|}{|H_0(j2\pi f_0)|} = \frac{\sin(\pi f_0/f_s)/\big(\pi(f_s - f_0)\big)}{\sin(\pi f_0/f_s)/(\pi f_0)} = \frac{f_0}{f_s - f_0}.$$

At $f_s = 2000$ Hz the copy is at 1700 Hz with relative level $300/1700 \approx 0.18$. At $f_s = 8000$ Hz it is at 7700 Hz with relative level $300/7700 \approx 0.039$.

CHECK The ratio is below 1 whenever $f_0 < f_s/2$, as it must be for a sampled tone. It falls as f_s rises: a higher rate moves the copies up and makes them weaker. The copies sound as a buzz on top of the tone, and raising the rate makes the buzz rise in pitch and fade.

Compensating the hold

The hold followed by a second filter H_r would act as the ideal filter H if $H_0(j\omega)\,H_r(j\omega) = H(j\omega)$. Divide by $H_0 = e^{-j\omega T/2}\,2\sin(\omega T/2)/\omega$:

$$H_r(j\omega) = \frac{H(j\omega)}{H_0(j\omega)} = \frac{e^{j\omega T/2}\,H(j\omega)\,\omega}{2\sin(\omega T/2)}.$$

Take the target H with gain T up to π/T and zero beyond. Inside the band the compensator needs $|H_r| = T\omega/\big(2\sin(\omega T/2)\big)$. At $\omega \to 0$ this is 1, since $\sin\theta \approx \theta$. At the edge $\omega = \pi/T$ it is $(T \cdot \pi/T)/\big(2\sin(\pi/2)\big) = \pi/2 \approx 1.57$. So the boost stays between 1 and $\pi/2$; it never grows without bound, because $\sin(\omega T/2)$ first vanishes at $\omega = \omega_s$, outside the band. A narrower target needs less boost: if H stops at $\omega_s/4 = \pi/(2T)$, the largest boost is $(\pi/2)/\big(2\sin(\pi/4)\big) = \pi/(2\sqrt{2}) \approx 1.11$.

The compensator is still hard to build. The factor $e^{j\omega T/2}$ is a time advance, and H is ideal. Converters approximate H_r, or use the hold output as it is.

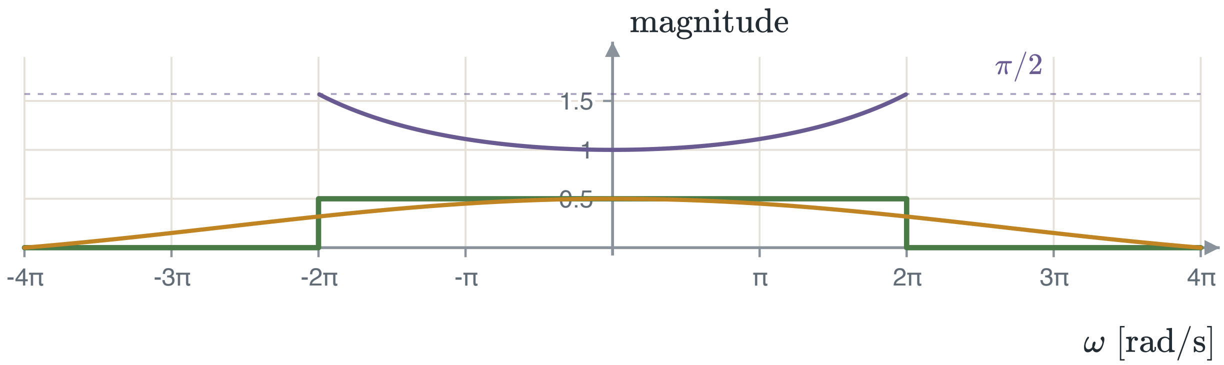

FIGURE 7.25 With $T = 0.5$ s: the hold $|H_0(j\omega)|$, amber, the compensator $|H_r(j\omega)|$, violet, which rises from 1 to $\pi/2$ at the band edge, and their product, green, the target of gain T up to $\omega_s/2 = 2\pi$ rad/s.

The first-order hold

The first-order hold joins consecutive samples by a straight line. It is also called linear interpolation.

$$x_1(t) = x(nT) + \frac{t - nT}{T}\big[x((n+1)T) - x(nT)\big], \qquad nT \le t < (n+1)T.$$

At $t = nT$ the fraction is 0 and $x_1 = x(nT)$; at $t = (n+1)T$ it is 1 and $x_1 = x((n+1)T)$. Between nT and $(n+1)T$ the line needs the next sample $x((n+1)T)$, so a real system waits for it and its output is $x_1(t)$ delayed by T.

The error is smaller than that of the staircase. Take one interval with midpoint m and expand x about m: $x(m \pm T/2) = x(m) \pm x'(m)\frac{T}{2} + x''(m)\frac{T^2}{8} + \cdots$. At the midpoint the line is the average of its two end values, $x(m) + x''(m)\,T^2/8 + \cdots$, because the first-order terms cancel. So the gap at the midpoint is about $|x''|\,T^2/8$ and scales with the square of the period: halving T from 0.2 s to 0.1 s about quarters it.

THE IMPULSE RESPONSE OF THE FIRST-ORDER HOLD

$$h_1(t) = \frac{1}{T}\,(g * g)(t), \qquad g(t) = \begin{cases} 1, & |t| \le T/2 \\ 0, & \text{otherwise} \end{cases}$$

$$(g * g)(t) = \int_{-\infty}^{\infty} g(\tau)\,g(t-\tau)\,\mathrm{d}\tau = \int_{t-T/2}^{T/2} 1\,\mathrm{d}\tau = \frac{T}{2} - \left(t - \frac{T}{2}\right) = T - t, \qquad 0 \le t \le T$$

g is the hold rectangle, centred on $t = 0$. The integrand is 1 where both $|\tau| \le T/2$ and $|t - \tau| \le T/2$. For $0 \le t \le T$ the second condition is $t - T/2 \le \tau \le t + T/2$, so the overlap is $t - T/2 \le \tau \le T/2$, of length $T - t$. For $t > T$ there is no overlap. The convolution is even, so $(g * g)(t) = T - |t|$ on $|t| \le T$: a triangle of peak T. Dividing by T gives $h_1(t) = 1 - |t|/T$ on $|t| \le T$, a triangle of peak 1. With $T = 0.2$ s, $h_1(0.1) = 1 - 0.1/0.2 = 0.5$.

This triangle joins the samples. The output of the hold is $\sum_k x(kT)\,h_1(t - kT)$. On $nT \le t \le (n+1)T$ only the kernels of n and $n+1$ are non-zero, and $|t - nT| = t - nT$, $|t - (n+1)T| = (n+1)T - t$:

$$\begin{aligned} &x(nT)\left(1 - \frac{t - nT}{T}\right) + x((n+1)T)\left(1 - \frac{(n+1)T - t}{T}\right) \\ &= x(nT) - x(nT)\,\frac{t - nT}{T} + x((n+1)T)\,\frac{t - nT}{T} = x(nT) + \frac{t - nT}{T}\big[x((n+1)T) - x(nT)\big], \end{aligned}$$

which is $x_1(t)$. The second bracket simplifies because $1 - \frac{(n+1)T - t}{T} = \frac{T - (n+1)T + t}{T} = \frac{t - nT}{T}$.

THE FREQUENCY RESPONSE OF THE FIRST-ORDER HOLD

$$G(j\omega) = \int_{-T/2}^{T/2} e^{-j\omega t}\,\mathrm{d}t = \left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-T/2}^{T/2} = \frac{e^{j\omega T/2} - e^{-j\omega T/2}}{j\omega} = \frac{2\sin(\omega T/2)}{\omega}$$

$$H_1(j\omega) = \frac{1}{T}\,G(j\omega)\,G(j\omega) = \frac{1}{T}\left[\frac{\sin(\omega T/2)}{\omega/2}\right]^2$$

G is the transform of the centred rectangle; the limits are $\pm T/2$ because g is zero outside them, and $e^{j\theta} - e^{-j\theta} = 2j\sin\theta$ cancels the j. Since g is centred on $t = 0$, G has no delay factor. Convolution in time is multiplication in frequency, so $g * g$ transforms to $G \cdot G$, and the constant $1/T$ passes through by linearity. H_1 is real and $H_1 \ge 0$ because it is a square. At $\omega = 0$, $\sin(\omega T/2)/(\omega/2) \to T$, so $H_1(j0) = T^2/T = T$.

At the band edge $\omega = \pi/T$: $H_1 = \frac{1}{T}\left[\sin(\pi/2)/(\pi/(2T))\right]^2 = \frac{1}{T}\cdot\frac{4T^2}{\pi^2} = \frac{4T}{\pi^2} \approx 0.41\,T$, a larger sag than the $0.64\,T$ of the zero-order hold. Far above the band H_1 falls off as $1/\omega^2$, against $1/\omega$ for $|H_0|$, so it leaks much less. With $T = 0.5$ s at $\omega = 6\pi$ rad/s, where $\omega T/2 = 1.5\pi$ and $|\sin(1.5\pi)| = 1$:

$$|H_0(j6\pi)| = \frac{2\cdot 1}{6\pi} = \frac{1}{3\pi} \approx 0.106, \qquad H_1(j6\pi) = \frac{1}{0.5}\left[\frac{1}{3\pi}\right]^2 = \frac{2}{9\pi^2} \approx 0.023.$$

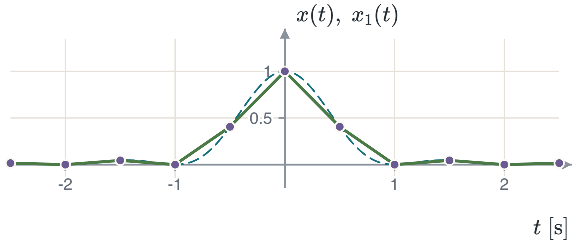

FIGURE 7.26 The first-order hold with $T = 0.5$ s: the samples of the running signal joined by straight lines.

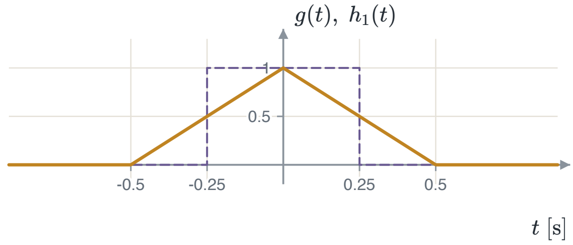

FIGURE 7.27 The hold rectangle $g(t)$, dashed, and the triangle $h_1(t) = (g * g)(t)/T$ of peak 1, with $T = 0.5$ s. The axis is time.

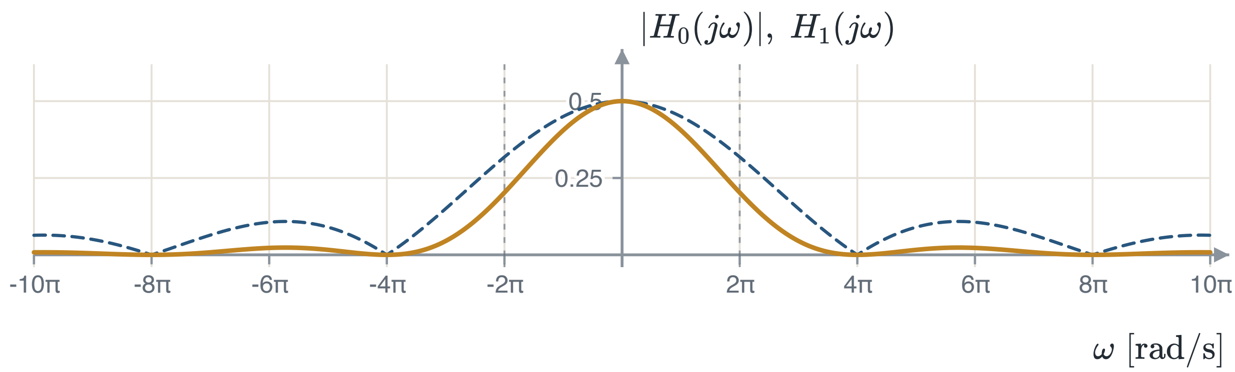

FIGURE 7.28 Both holds with $T = 0.5$ s: the first-order hold, solid amber, and the zero-order hold, dashed. The band edge $\omega_s/2 = 2\pi$ rad/s is marked. Both start at T; the first-order hold falls away faster on both sides of the edge.

A HOLD IS NOT A RECONSTRUCTION

Both holds are filters with a real, non-flat, non-band-limited response. The staircase and the broken line are different signals from $x(t)$, and a faster rate makes the error smaller but never zero. The sampling theorem guarantees that the samples *determine* $x(t)$ and names the filter that recovers it. Feeding the same samples to a different filter gives a different output.

Conditions for exact reconstruction

Exact reconstruction needs three things together.

1. **Band-limited**: $X(j\omega) = 0$ for $|\omega| > \omega_M$.
2. **Rate**: $\omega_s > 2\omega_M$.
3. **Filter**: gain T and cutoff $\omega_M < \omega_c < \omega_s - \omega_M$.

The first condition fails for every signal of finite duration. Take the rectangular pulse $x(t) = 1$ on $|t| \le T_1$ and 0 outside:

$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t}\,\mathrm{d}t = \left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-T_1}^{T_1} = \frac{e^{j\omega T_1} - e^{-j\omega T_1}}{j\omega} = \frac{2\sin(\omega T_1)}{\omega}.$$

This transform crosses zero again and again, but it is never zero on a whole interval. There is no ω_M beyond which it vanishes, so no rate keeps the copies apart, and the tails of all the copies add everywhere. In practice a filter first removes what lies above a chosen ω_M. The filtered signal is then recovered exactly; the original is not. By contrast, $\cos(10\pi t)$ has $\omega_M = 10\pi$ rad/s, so any $\omega_s > 20\pi$ rad/s recovers it exactly.

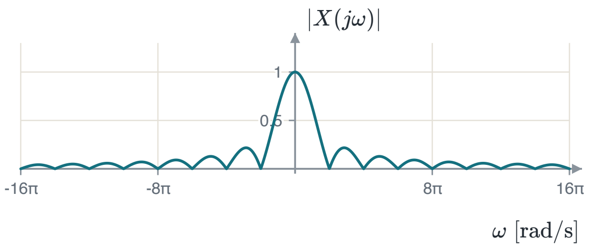

FIGURE 7.29 A rectangular pulse of width 1 s, $T_1 = 0.5$ s, has $|X(j\omega)| = |2\sin(\omega/2)/\omega|$, which never settles to zero.

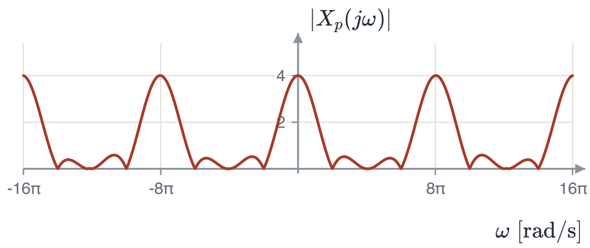

FIGURE 7.30 The same pulse sampled with $T = 0.25$ s, $\omega_s = 8\pi$ rad/s: the tails of all the copies add at every frequency.

7.4 Aliasing in practice

Aliasing of a sampled cosine

Take $x(t) = \cos(\omega_0 t)$, so $\omega_M = \omega_0$, and reconstruct with the cutoff at $\omega_c = \omega_s/2$. The transform is $X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$. Substitute it into the sampled-spectrum sum:

$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\big(j(\omega - k\omega_s)\big) = \frac{\pi}{T}\sum_{k=-\infty}^{\infty}\big[\delta(\omega - k\omega_s - \omega_0) + \delta(\omega - k\omega_s + \omega_0)\big].$$

So sampling puts a line at every $k\omega_s \pm \omega_0$, each of weight π/T. The filter multiplies by T inside $|\omega| < \omega_c$ and by 0 outside, so each surviving line returns to weight π. The method has three steps: list the lines $k\omega_s \pm \omega_0$, keep those with $|\omega| < \omega_c$, and read the output from the kept lines. For $k = 0$ and $k = 1$ at three rates (the $k = -1$ lines are the mirror images at negative frequencies and give the same verdict):

$$\begin{aligned}
\omega_s = 6\omega_0,\ \omega_c = 3\omega_0 :&\quad k = 0:\ \pm\omega_0 \text{ inside};\quad k = 1:\ 6\omega_0 \pm \omega_0 = 5\omega_0,\ 7\omega_0 \text{ outside}\\
&\quad \Rightarrow\ x_r(t) = \cos(\omega_0 t)\\
\omega_s = 3\omega_0,\ \omega_c = 1.5\omega_0 :&\quad k = 0:\ \pm\omega_0 \text{ inside};\quad k = 1:\ 3\omega_0 \pm \omega_0 = 2\omega_0,\ 4\omega_0 \text{ outside}\\
&\quad \Rightarrow\ x_r(t) = \cos(\omega_0 t)\\
\omega_s = 1.5\omega_0,\ \omega_c = 0.75\omega_0 :&\quad k = 0:\ \pm\omega_0 \text{ outside};\quad k = 1:\ 0.5\omega_0 \text{ inside},\ 2.5\omega_0 \text{ outside}\\
&\quad \Rightarrow\ x_r(t) = \cos(0.5\omega_0 t)
\end{aligned}$$

In the third case the original frequency is removed: the line at ω_0 lies outside the filter. The line at $1.5\omega_0 - \omega_0 = 0.5\omega_0$ comes from the $k = 1$ copy, and it lies inside.

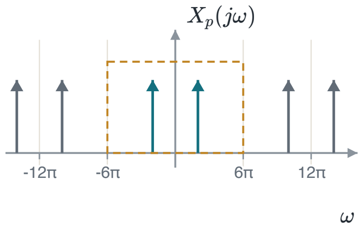

FIGURE 7.31 $\omega_s = 6\omega_0$: the line at ω_0 is kept.

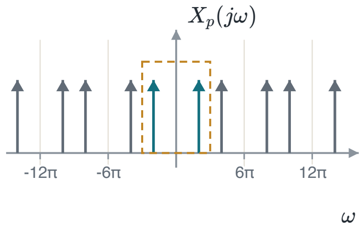

FIGURE 7.32 $\omega_s = 3\omega_0$: the line at ω_0 is still kept.

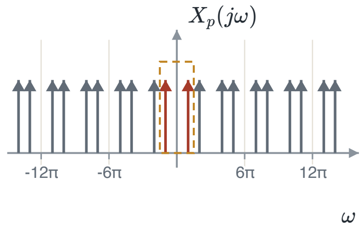

FIGURE 7.33 $\omega_s = 1.5\omega_0$: a copy line at $0.5\omega_0$ is kept, red; the others are grey.

The alias frequency

THE UNDERSAMPLED COSINE, $\omega_s/2 < \omega_0 < \omega_s$

$$X_r(j\omega) = \pi\delta\big(\omega - (\omega_s - \omega_0)\big) + \pi\delta\big(\omega + (\omega_s - \omega_0)\big) \xrightarrow{\mathcal{F}^{-1}} x_r(t) = \cos\big((\omega_s - \omega_0)t\big)$$

For $\omega_s/2 < \omega_0 < \omega_s$ the baseband line at ω_0 lies outside $|\omega| < \omega_s/2$, while the line $\omega_s - \omega_0$ of the copy $k = 1$ lies inside, and so does its mirror $-(\omega_s - \omega_0)$ from $k = -1$. Each impulse is written as a function of ω ; an expression such as $\pi\delta(\omega_s - \omega_0)$ is a constant and locates nothing. The arrow is the inverse transform.

The output is a clean cosine at the wrong frequency, and nothing in the samples marks it. The samples of $\cos(\omega_0 t)$ and of $\cos\big((\omega_s - \omega_0)t\big)$ are the same numbers, because $(\omega_s - \omega_0)nT = 2\pi n - \omega_0 nT$ and the cosine is even with period 2π.

For any ω_0 the rule reads as follows. The numbers $\omega_0 - k\omega_s$, for all integers k, are spaced ω_s apart, so exactly one of them lies in $-\omega_s/2 < \omega < \omega_s/2$ (when none sits on an end); it is the one with k the integer nearest to ω_0/ω_s. Its mirror comes from the negative-frequency line. So the kept pair is $\pm|\omega_0 - k\omega_s|$. In hertz the tone returns at

$$f_a = |f_0 - kf_s|, \qquad k = \text{the integer nearest to } f_0/f_s, \qquad 0 \le f_a \le f_s/2.$$

The simpler form $\omega_s - \omega_0$ is the case $k = 1$, which holds for $\omega_s/2 < \omega_0 < 3\omega_s/2$ only. For example, at $f_s = 6$ kHz a tone at $f_0 = 4$ kHz returns at $|4 - 6| = 2$ kHz, so a 4 kHz whistle recorded this way plays back at 2 kHz. A tone at 5 kHz returns at $|5 - 6| = 1$ kHz, the only line inside $|f| < 3$ kHz.

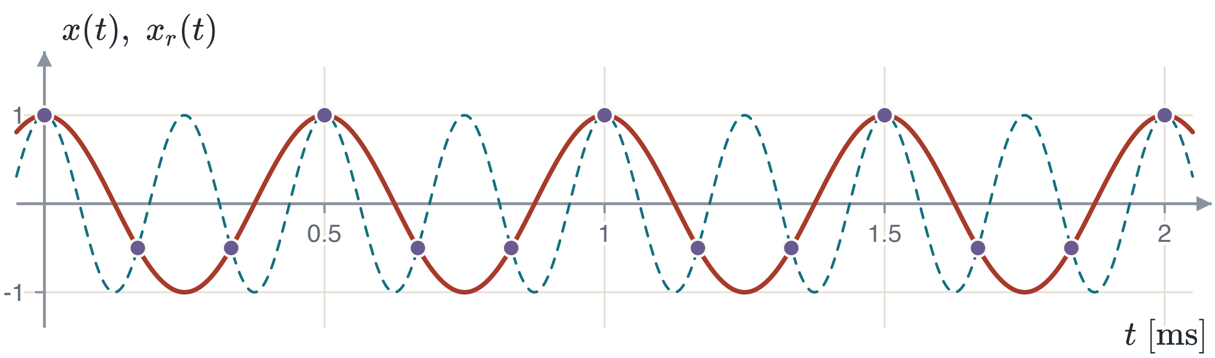

FIGURE 7.34 A 4 kHz tone, dashed, sampled at $f_s = 6$ kHz, violet dots. The red 2 kHz cosine passes through the same samples, and it is what the filter returns.

Hearing aliasing with a chirp

A **chirp** is a tone whose frequency rises steadily. Take $x(t) = \cos\left(2\pi \cdot 1000\, t^2\right)$. The frequency at a moment is the rate of change of the phase, divided by 2π:

$$f(t) = \frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}t}\left(2\pi \cdot 1000\, t^2\right) = \frac{1}{2\pi} \cdot 2\pi \cdot 2000\, t = 2000\, t \text{ Hz}.$$

In 3 s it rises from 0 to 6 kHz. Sample it at $f_s = 4$ kHz and apply the fold rule at each moment. For $0 \le t \le 1$ s, $f \le 2$ kHz $= f_s/2$ and $k = 0$: the tone is heard as it is, rising to 2 kHz at $t = 1$ s. For $1 \le t \le 2$ s, $2 \le f \le 4$ kHz and $k = 1$: the heard pitch is $4000 - 2000t$ Hz, which falls to 0 at $t = 2$ s, where $f = f_s$. For $2 \le t \le 3$ s, $4 \le f \le 6$ kHz and again $k = 1$: the heard pitch is $2000t - 4000$ Hz, which rises again to 2 kHz at $t = 3$ s. The pitch that comes out folds at $f_s/2$ and runs back down.

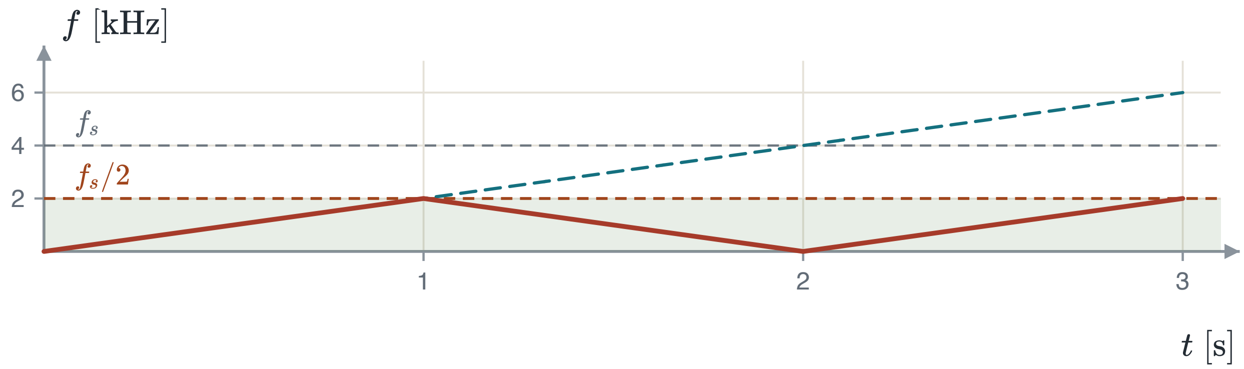

FIGURE 7.35 The chirp frequency $f(t) = 2000\, t$ Hz, dashed, sampled at $f_s = 4$ kHz, and the pitch that comes out, red. The green band is what the reconstruction filter keeps.

EXAMPLE 7.8 — ALIASING AT THREE SAMPLING PERIODS

GIVEN $x(t) = \cos(2\pi t)$, so $\omega_M = 2\pi$ rad/s and the Nyquist rate is 4π rad/s. The periods are $T_1 = 1/4$ s, $T_2 = 1/3$ s and $T_3 = 2/3$ s, and $\omega_c = \omega_s/2$.

FIND Which periods recover the signal, and what is recovered when one does not.

METHOD Use the line spectrum after sampling, because the filter selects individual lines. Compute $\omega_s = 2\pi/T$, compare it with the Nyquist rate of **this** signal, 4π, and when it fails find the line inside $|\omega| < \omega_s/2$.

SOLUTION

$$\begin{aligned} T_1 = 1/4\text{ s}: &\quad \omega_s = \frac{2\pi}{1/4} = 8\pi > 4\pi, \quad \omega_c = 4\pi \\ T_2 = 1/3\text{ s}: &\quad \omega_s = \frac{2\pi}{1/3} = 6\pi > 4\pi, \quad \omega_c = 3\pi \\ T_3 = 2/3\text{ s}: &\quad \omega_s = \frac{2\pi}{2/3} = 3\pi < 4\pi, \quad \omega_c = 1.5\pi \end{aligned}$$

In the first two rows the baseband lines $\pm 2\pi$ are inside the cutoff, and the nearest copy lines, $8\pi - 2\pi = 6\pi$ and $6\pi - 2\pi = 4\pi$, are outside, so $x_r(t) = \cos(2\pi t)$. In the third row the baseband lines $\pm 2\pi$ lie outside $|\omega| < 1.5\pi$ and are removed. The $k = 1$ copy puts lines at $3\pi \pm 2\pi$, that is at 5π and at π; the one at π is inside. The $k = -1$ copy gives the mirror line at $-\pi$. So

$$\omega_s - \omega_0 = 3\pi - 2\pi = \pi < \omega_c = 1.5\pi,$$
$$X_r(j\omega) = \pi\delta(\omega - \pi) + \pi\delta(\omega + \pi) \implies x_r(t) = \cos(\pi t).$$

CHECK The sample values of the two signals must agree at $T_3 = 2/3$ s:

$$\cos(2\pi n T_3) = \cos\left(\frac{4\pi n}{3}\right) = \cos\left(2\pi n - \frac{2\pi n}{3}\right)$$
$$= \cos\left(\frac{2\pi n}{3}\right) = \cos(\pi n T_3).$$

Here $2\pi n$ is a whole number of turns, and the cosine is even. The two sample sequences are identical, so no reconstruction could prefer one over the other; the filter returns the lower one. At $T = 0.8$ s, similarly, $\omega_s = 2\pi/0.8 = 2.5\pi$ and $\omega_c = 1.25\pi$: the line 2π is outside, and the copy line $2.5\pi - 2\pi = 0.5\pi$ is inside, so $x_r(t) = \cos(0.5\pi t)$.

WARNING The verdict rests on the number the rate is compared with, and for this signal that number is $2\omega_M = 4\pi$. Comparing against another signal's Nyquist rate of 6π makes the middle row read $6\pi \geq 6\pi$, the boundary case the theorem excludes, although this signal has a comfortable margin there.

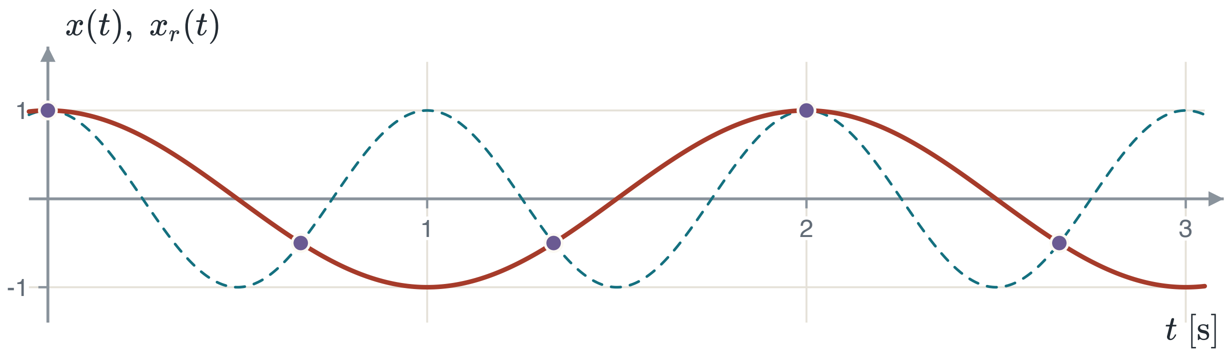

FIGURE 7.36 Example 7.8 at $T_3 = 2/3$ s: $\cos(2\pi t)$, dashed, its samples, and the red $\cos(\pi t)$ that the filter returns through the same samples.

EXAMPLE 7.9 – PARTIAL ALIASING OF TWO COSINES

GIVEN $x(t) = \cos(\pi t) + \cos(3\pi t)$, sampled with $T = 2/5$ s; $\omega_c = \omega_s/2$.

FIND $x_r(t)$, and which component moved.

METHOD Treat the components separately, because one can alias while the other passes unchanged. First test the rate against this signal's Nyquist rate, then track each baseband and copy line through the filter.

SOLUTION $\omega_M = 3\pi$ rad/s, so the Nyquist rate is $2\omega_M = 6\pi$ rad/s, while

$$\omega_s = \frac{2\pi}{T} = \frac{2\pi}{2/5} = 5\pi\ \mathrm{rad/s}, \qquad \omega_c = \frac{\omega_s}{2} = 2.5\pi\ \mathrm{rad/s}.$$

Since $5\pi < 6\pi$, aliasing occurs. Each component transforms to a pair of impulses of weight π, and sampling puts lines at $k\omega_s \pm \omega_0$ for each of the two values of ω_0. Test each line against $|\omega| < 2.5\pi$:

$$
\begin{aligned}
\omega_0 = \pi,\ k = 0:&\quad \pm\pi \text{ inside}\\
\omega_0 = \pi,\ k = 1:&\quad 5\pi - \pi = 4\pi,\ 5\pi + \pi = 6\pi \text{ outside}\\
\omega_0 = 3\pi,\ k = 0:&\quad \pm 3\pi \text{ outside}\\
\omega_0 = 3\pi,\ k = 1:&\quad 5\pi - 3\pi = 2\pi \text{ inside},\ 5\pi + 3\pi = 8\pi \text{ outside}
\end{aligned}
$$

Larger $|k|$ moves the lines further out, and $k = -1$ gives the mirror lines at -2π and -4π. The filter multiplies the surviving lines by T, which cancels the $1/T$ of sampling and returns the weight π to each line. Hence

$$
\begin{aligned}
X_r(j\omega) &= \pi\big[\delta(\omega-\pi)+\delta(\omega+\pi)\big] + \pi\big[\delta(\omega-2\pi)+\delta(\omega+2\pi)\big]\\
\xrightarrow{\mathcal{F}^{-1}}\ x_r(t) &= \cos(\pi t) + \cos(2\pi t),
\end{aligned}
$$

since each pair $\pi[\delta(\omega-a)+\delta(\omega+a)]$ returns $\cos(at)$. The component $\cos(\pi t)$ keeps its frequency; $\cos(3\pi t)$ returns at $\omega_s - 3\pi = 2\pi$, a frequency the signal never contained.

CHECK The aliased component and its replacement must give the same samples at $T = 2/5$ s:

$$
\begin{aligned}
\cos(3\pi nT) &= \cos\left(\frac{6\pi n}{5}\right) = \cos\left(2\pi n - \frac{4\pi n}{5}\right)\\
&= \cos\left(\frac{4\pi n}{5}\right) = \cos(2\pi nT).
\end{aligned}
$$

VARIATION With $T = 1/2$ s, $\omega_s = 4\pi$ and $\omega_c = 2\pi$. The line π is kept. The copy line of the second component lands at $4\pi - 3\pi = \pi$, on top of the kept line, so the two weights add: $X_r(j\omega) = 2\pi\big[\delta(\omega-\pi)+\delta(\omega+\pi)\big]$ and $x_r(t) = 2\cos(\pi t)$. Saying only "aliasing occurred" is not an answer; name what each component became.

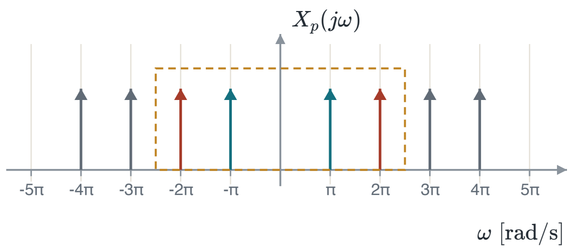

FIGURE 7.37 The lines of Example 7.9 after sampling at $\omega_s = 5\pi$ rad/s, with the filter $|\omega| < 2.5\pi$ dashed. The line at π is kept, cyan, the copy line at 2π is kept, red, and the grey lines are removed.

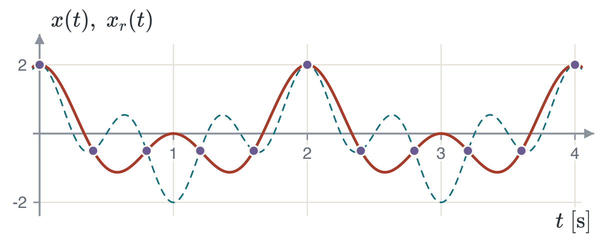

FIGURE 7.38 $x(t)$, dashed, and the red $x_r(t) = \cos(\pi t) + \cos(2\pi t)$. Both pass through every sample.

The anti-aliasing filter

No real signal is strictly band-limited, and a sampler at a fixed rate cannot be helped after the fact. The repair is a low-pass filter H_{AA} placed **before** the sampler, which removes everything above $\omega_s/2$ while it can still be removed cleanly.

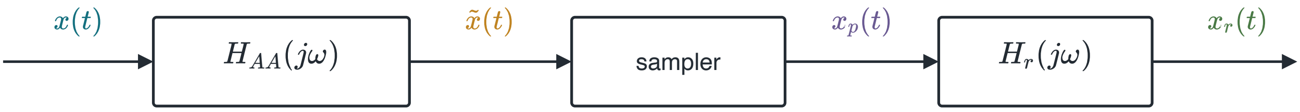

FIGURE 7.39 The chain with an anti-aliasing filter. The filter comes first; the sampler output is $x_p(t)$, and $x_r(t)$ is the output of the reconstruction filter.

> **PLACE THE FILTER BEFORE THE SAMPLER**
>
> After sampling, a folded frequency sits on top of wanted content at the same frequency. No later filter can separate the two contributions. The anti-aliasing filter must remove the high frequencies before the sampler.

EXAMPLE 7.10 – WHAT THE ANTI-ALIASING FILTER BUYS

GIVEN $X(j\omega) = 1 - |\omega|/(3\pi)$ on $|\omega| \le 3\pi$ and 0 outside, a triangle that reaches past $\omega_s/2$ for $\omega_s = 4\pi$ rad/s. Before the sampler, an ideal low-pass H_{AA} keeps $|\omega| < \omega_a$; the reconstruction filter has gain T and cutoff $\omega_s/2 = 2\pi$.

FIND The error energy $\frac{1}{2\pi}\int |X(j\omega) - X_r(j\omega)|^2\,\mathrm{d}\omega$ for $\omega_a = 3\pi$ (no filter), $\omega_a = 2\pi$ and $\omega_a = \pi$.

METHOD By Parseval this is the energy of $x(t) - x_r(t)$. Split the frequency axis into the part the reconstruction keeps, $|\omega| < 2\pi$, and the part it removes, $2\pi < |\omega| < 3\pi$. Inside the band the error is whatever the copies add or the prefilter removed; outside it the error is X itself. The triangle is even, so compute $\omega > 0$ and double.

SOLUTION No filter, $\omega_a = 3\pi$. For $0 < \omega < 2\pi$ the copy $k = 1$ adds $X(j(\omega - 4\pi))$, which is non-zero where $|\omega - 4\pi| \le 3\pi$, that is for $\omega \ge \pi$; there it equals $1 - (4\pi - \omega)/(3\pi) = (\omega - \pi)/(3\pi)$. So the error inside the band is $(\omega - \pi)/(3\pi)$ on $\pi < \omega < 2\pi$, and outside it is $X = (3\pi - \omega)/(3\pi)$ on $2\pi < \omega < 3\pi$. With $u = \omega - \pi$ in the first integral and $u = 3\pi - \omega$ in the second, each runs over $0 < u < \pi$:

$$\int_{\pi}^{2\pi} \left(\frac{\omega - \pi}{3\pi}\right)^2 \mathrm{d}\omega = \int_{2\pi}^{3\pi} \left(\frac{3\pi - \omega}{3\pi}\right)^2 \mathrm{d}\omega = \frac{1}{9\pi^2}\left[\frac{u^3}{3}\right]_0^{\pi} = \frac{\pi}{27}$$

$$E_{3\pi} = \frac{1}{2\pi}\cdot 2\left(\frac{\pi}{27} + \frac{\pi}{27}\right) = \frac{2}{27} \approx 0.074.$$

With $\omega_a = 2\pi$ the prefilter removes $2\pi < |\omega| < 3\pi$ before the sampler, so no copy reaches the band and only the removed part is lost:

$$E_{2\pi} = \frac{1}{2\pi}\cdot 2 \cdot \frac{\pi}{27} = \frac{1}{27} \approx 0.037.$$

With $\omega_a = \pi$ the whole of $\pi < |\omega| < 3\pi$ is lost; with $u = 3\pi - \omega$ running over $0 < u < 2\pi$:

$$E_\pi = \frac{1}{2\pi} \cdot 2 \int_\pi^{3\pi} \left(\frac{3\pi - \omega}{3\pi}\right)^2 \mathrm{d}\omega = \frac{1}{\pi} \cdot \frac{1}{9\pi^2} \left[\frac{u^3}{3}\right]_0^{2\pi} = \frac{1}{\pi} \cdot \frac{8\pi^3}{27\pi^2} = \frac{8}{27} \approx 0.296.$$

CHECK

Moving ω_a from 3π to 2π removes the folded part and halves the error: $\frac{1/27}{2/27} = \frac{1}{2}$. A cutoff below $\omega_s/2$ removes wanted content too, with no folded part left to remove, so $E_\pi = 8/27$ is larger than both. The best cutoff is $\omega_s/2$.

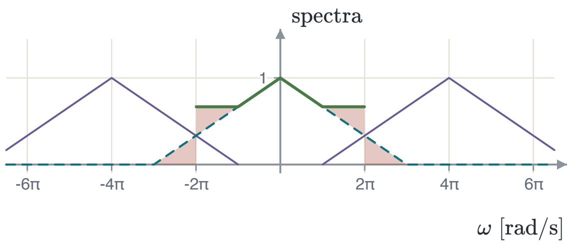

FIGURE 7.40 No anti-aliasing filter: $X(j\omega)$, dashed, the copies at $\pm 4\pi$, violet, and the recovered $X_r(j\omega)$, green. The red area is the error; its energy is $2/27$.

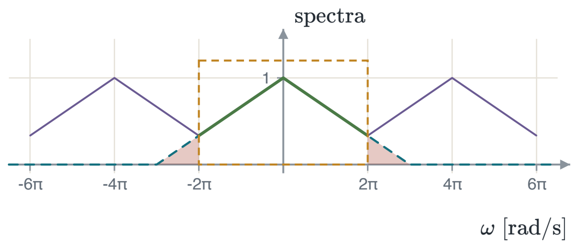

FIGURE 7.41 With H_{AA} cutting at $\omega_a = 2\pi$, dashed amber: the copies no longer reach the band, and only the removed tails are lost. The error energy halves to $1/27$.

The same comparison for the two cosines of Example 7.9, $x(t) = \cos(\pi t) + \cos(3\pi t)$ at $T = 2/5$ s. Without the filter, $x_r(t) = \cos(\pi t) + \cos(2\pi t)$ and the error is $e(t) = x(t) - x_r(t) = \cos(3\pi t) - \cos(2\pi t)$. With a low-pass at 2.5π ahead of the sampler, the component at 3π is removed first, $\cos(\pi t)$ alone is sampled and recovered exactly, and the error is $e(t) = \cos(3\pi t)$.

MEAN-SQUARE ERROR, AVERAGED OVER ONE PERIOD OF 2 S

$$\begin{aligned} e^2(t) &= \left[\cos(3\pi t) - \cos(2\pi t)\right]^2 = \cos^2(3\pi t) - 2\cos(3\pi t)\cos(2\pi t) + \cos^2(2\pi t) \\ &= \cos^2(3\pi t) - \cos(\pi t) - \cos(5\pi t) + \cos^2(2\pi t) \end{aligned}$$

$$\overline{e^2}\Big|_{\text{no filter}} = \frac{1}{2}\int_0^2 e^2(t)\,\mathrm{d}t = \tfrac{1}{2} - 0 - 0 + \tfrac{1}{2} = 1, \qquad \overline{e^2}\Big|_{\text{filtered}} = \frac{1}{2}\int_0^2 \cos^2(3\pi t)\,\mathrm{d}t = \tfrac{1}{2}$$

The cross term uses $2\cos A \cos B = \cos(A - B) + \cos(A + B)$ with $A = 3\pi t$ and $B = 2\pi t$. The window of 2 s holds a whole number of periods of every term: $\cos(\pi t)$ has period 2 s, $\cos(5\pi t)$ has period 0.4 s, and $\cos^2(3\pi t) = \frac{1}{2} + \frac{1}{2}\cos(6\pi t)$ averages to $\frac{1}{2}$. A cosine averages to zero over a whole number of periods, so each cross term contributes 0 and each squared cosine contributes $\frac{1}{2}$. Filtering first halves the error power. It costs the 3π component, which was lost either way. What it buys is the removal of a false component at 2π that the signal never contained, and that nothing downstream could tell apart from the signal.

Why 44.1 kHz

Hearing ends near 20 kHz, so the anti-aliasing filter of an audio converter passes 0 to 20 kHz. A real filter cannot drop at once: it needs a width Δ, its **transition band**, to fall to its stop level at $20 + \Delta$ kHz. After the filter the spectrum occupies $|f| \leq 20 + \Delta$ kHz. Its copy at f_s begins to rise at $f_s - (20 + \Delta)$, and it must not start before the filter has stopped:

$$f_s - (20 + \Delta) \geq 20 + \Delta \quad \Longrightarrow \quad f_s \geq 2(20 + \Delta) = 40 + 2\Delta \text{ kHz}.$$

EXAMPLE 7.11 – THE TRANSITION BAND OF THREE RATES

GIVEN An audio band of 0 to 20 kHz, and the rates 40 kHz, 44.1 kHz (the CD) and 48 kHz (film and video sound).

FIND The transition width Δ each rate leaves the anti-aliasing filter, and the lowest rate for a filter that needs $\Delta = 3$ kHz.

METHOD Solve $f_s = 2(20 + \Delta)$ for Δ, which gives $\Delta = f_s/2 - 20$.

SOLUTION

$$\begin{aligned} f_s = 40: &\quad \Delta = 20 - 20 = 0 \\ f_s = 44.1: &\quad \Delta = 22.05 - 20 = 2.05\ \mathrm{kHz} \\ f_s = 48: &\quad \Delta = 24 - 20 = 4\ \mathrm{kHz} \end{aligned}$$

40 kHz would need $\Delta = 0$, a filter no circuit can build. The CD rate leaves 2.05 kHz, from 20 to 22.05 kHz. A cheaper filter that needs $\Delta = 3$ kHz requires $f_s \geq 2(20 + 3) = 46$ kHz; at 44.1 kHz its copies would overlap.

CHECK Put each Δ back: $40 + 2 \times 2.05 = 44.1$ and $40 + 2 \times 4 = 48$. The same rule serves a telephone line, which keeps speech up to 3.4 kHz and samples at 8 kHz: $\Delta = 8/2 - 3.4 = 0.6$ kHz.

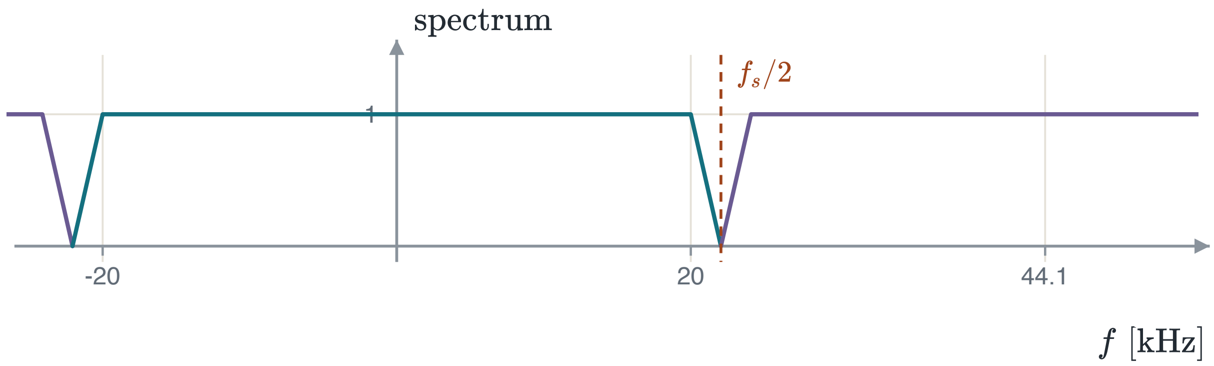

FIGURE 7.42 A flat input after the anti-aliasing filter, cyan, and its copies at the CD rate $f_s = 44.1$ kHz, violet. With $\Delta = 2.05$ kHz the filter reaches its stop level exactly where the copy at f_s begins, at $f_s/2 = 22.05$ kHz.

Aliasing in time

A camera is a sampler with $T = 1/f_s$, where f_s is the frame rate, and a strobe lamp is the same device built from light. What is seen follows the alias rule, applied to an angle.

EXAMPLE 7.12 – A WHEEL THAT TURNS BACKWARDS

GIVEN A marked spoke turns at 9 revolutions per second. A camera records 10 frames per second.

FIND The rotation the recording shows, with its direction.

METHOD Model the tip of the spoke as $e^{j\theta(t)}$ with $\theta(t) = 2\pi \cdot 9\,t$, a single complex exponential, so that the direction of rotation is the sign of its frequency. Sample it at $\omega_s = 2\pi f_s$ and keep the line inside $|\omega| < \omega_s/2$.

SOLUTION The tip $e^{j18\pi t}$ has one line, at $\omega_0 = 2\pi \cdot 9 = 18\pi$ rad/s. The camera samples at $f_s = 10$ Hz, so $\omega_s = 20\pi$ rad/s and the kept band is $|\omega| < 10\pi$. Sampling puts copies of the line at

$18\pi + k \cdot 20\pi$:

$$\ldots,\ -22\pi,\ \underbrace{-2\pi}_{k=-1},\ \underbrace{18\pi}_{k=0},\ 38\pi,\ \ldots$$

Only $k = -1$ lies inside, so the kept line is $\omega_0 - \omega_s = 18\pi - 20\pi = -2\pi$ rad/s. That is $2\pi/2\pi = 1$ revolution per second, and the minus sign means the other direction. Since $\omega_s = 20\pi < 2\omega_0 = 36\pi$, this is the undersampled case. Between frames the spoke turns $9/10$ of a turn, and the eye reads that as $1/10$ of a turn backwards, because it takes the shorter way round.

CHECK Frame n is taken at $t = n/10$ s. The recorded angle is

$$\theta\left(\frac{n}{10}\right) = 18\pi\,\frac{n}{10} = 2\pi n - \frac{2\pi n}{10},$$

and $e^{j(2\pi n - 2\pi n/10)} = e^{-j2\pi n/10}$ because $e^{j2\pi n} = 1$. These are exactly the frames of a rotation at -1 revolution per second. At 20 frames per second, $\omega_s = 40\pi > 36\pi$, and the film shows 9 rev/s in the right direction. At 9 frames per second the kept line is $18\pi - 18\pi = 0$, and the spoke stands still. A fan at 25 rev/s filmed at 24 frames per second gives $50\pi - 48\pi = +2\pi$: 1 rev/s forward.

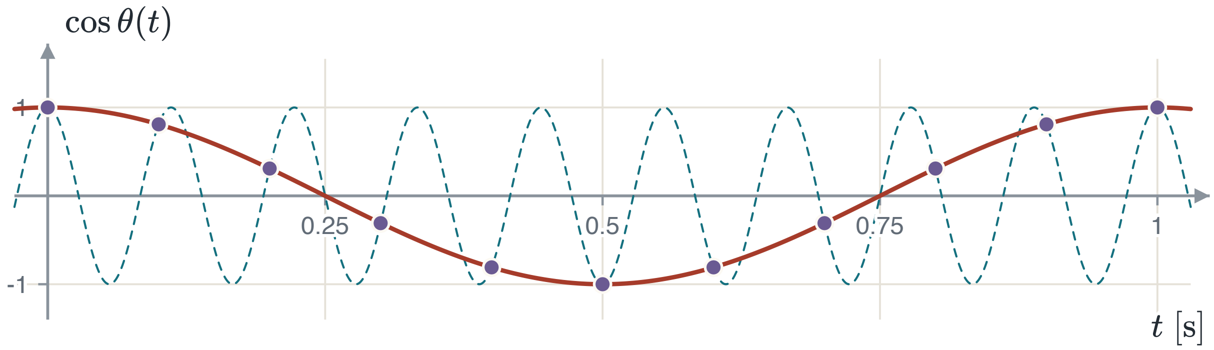

FIGURE 7.43 The horizontal position of the spoke tip, $\cos\theta(t)$, at 9 revolutions per second, dashed, and the frames at 10 per second, dots. The frames describe one revolution per second, red.

Aliasing in space

Position takes the place of time without any change to the arithmetic. A stripe pattern of 9 cycles per millimetre recorded on a grid of 10 points per millimetre has $2 \times 9 = 18 > 10$, so it is undersampled, and it is recorded at $|10 - 9| = 1$ cycle per millimetre. A fine pattern is recorded as a coarse one. The coarse bands are in the samples, not in the object. They are smooth and regular, and nothing marks them as false. The cure is the one used in time: blur the detail slightly before it is recorded, not after.

Two regular patterns laid over each other make the same effect with no sampler at all. Stripes of period 1.0 mm and 1.1 mm have spatial frequencies

$$f_1 = \frac{1}{1.0} = 1.000, \qquad f_2 = \frac{1}{1.1} = 0.909\ \text{cycles/mm}, \qquad P = \frac{1}{f_1 - f_2} = \frac{1}{1 - \frac{1}{1.1}} = \frac{1.1}{0.1} = 11\ \text{mm}.$$

Two close spatial frequencies beat at their difference, as two close tones do, and the overlay repeats every P. This pattern is called **moiré**. When an image is recorded, the pixel grid is one of the two patterns:

that is the moiré on a photographed screen.

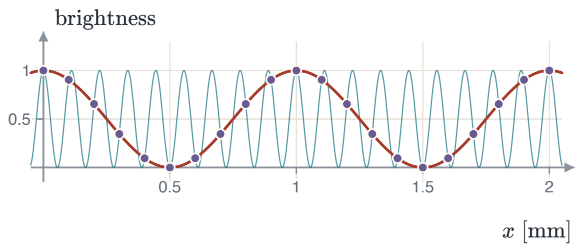

FIGURE 7.44 Stripes of 9 cycles per millimetre, cyan, recorded by a grid of 10 points per millimetre, dots. The grid records 1 cycle per millimetre, red.

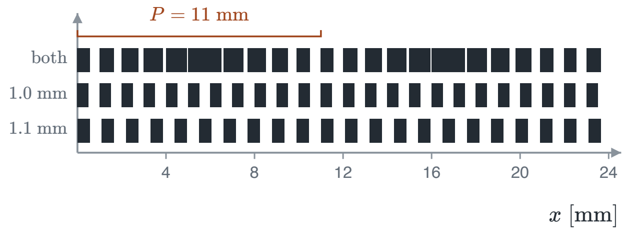

FIGURE 7.45 Bars of period 1.0 mm and 1.1 mm, alone and laid over each other. The overlay is dark where the bars disagree, and that repeats every $P = 11$ mm.

7.5 Discrete-time processing

Most signal processing today samples a continuous-time signal, computes with the numbers, and converts the result back. This section follows a band-limited signal through that chain and shows that, for such inputs, the whole chain acts as one continuous-time LTI system.

> ⓘ TWO FREQUENCY VARIABLES
>
> Both kinds of frequency appear together here, so this section writes ω for continuous-time frequency in rad/s, with $X_c(j\omega)$, and Ω for discrete-time frequency in rad/sample, with $X_d(e^{j\Omega}) = \sum_n x_d[n]e^{-j\Omega n}$. Chapter 6 wrote ω for the discrete-time variable; Section 7.6 returns to that. The subscript c marks a continuous-time signal and d a sequence.

The processing chain

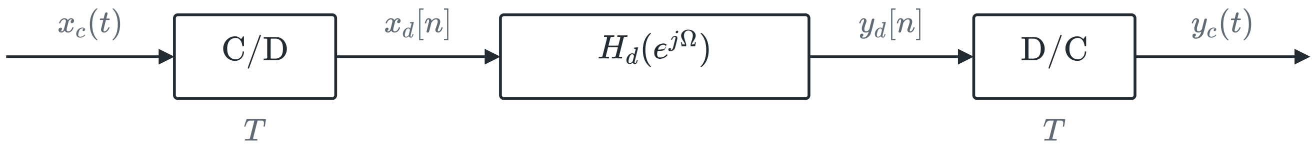

FIGURE 7.46 Discrete-time processing of a continuous-time signal: a C/D converter, a discrete-time system and a D/C converter, both converters working with the same period T.

$$x_d[n] = x_c(nT), \qquad y_c(t) = \sum_{n=-\infty}^{\infty} y_d[n]\,\mathrm{sinc}\left(\frac{\pi(t-nT)}{T}\right), \qquad \mathrm{sinc}(\theta) = \frac{\sin\theta}{\theta}.$$

The **C/D** (continuous-to-discrete) converter keeps the samples. The **D/C** converter rebuilds the band-limited signal through the output numbers, with the ideal filter of gain T and cutoff π/T of Section 7.3, so that $y_c(nT) = y_d[n]$ at every sample. Only H_d changes the spectrum; the converters sample and rebuild. If $y_d[n] = x_d[n]$ and the input is band-limited with $\omega_s > 2\omega_M$, the chain returns $y_c(t) = x_c(t)$.

From radians per second to radians per sample

Sample n is taken at $t = nT$, so a continuous-time exponential $e^{j\omega t}$ becomes $e^{j\omega nT} = e^{j(\omega T)n}$. Its discrete-time frequency is therefore

FREQUENCY MAP

$$\underbrace{\Omega}_{\text{rad/sample}} = \underbrace{\omega}_{\text{rad/s}} \underbrace{T}_{\text{s/sample}}$$

T is seconds per sample, so the units match. The sampling rate $\omega_s = 2\pi/T$ lands on $\Omega = \omega_s T = 2\pi$, and $\omega_s/2$ lands on $\Omega = \pi$. The band edge ω_M lands on $\Omega_M = \omega_M T$. The copies stay apart while $\Omega_M < \pi$, that is $\omega_M T < \pi$, that is $\omega_M < \pi/T = \omega_s/2$: the sampling theorem again.

For the running signal, $\omega_M = 2\pi$ rad/s, sampled with $T = 0.2$ s, the band edge lands on $\Omega_M = 2\pi \times 0.2 = 0.4\pi$. With $T = 0.25$ s it lands on 0.5π. With $T = 0.6$ s it lands on $1.2\pi > \pi$, and the copies overlap.

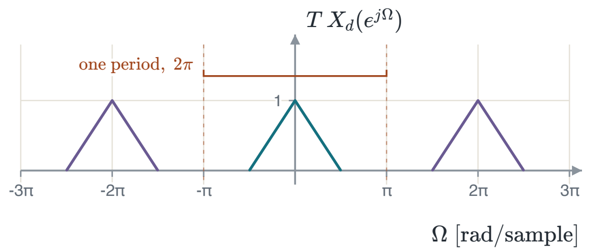

FIGURE 7.47 The running signal sampled with $T = 0.25$ s and drawn times T against Ω: $\Omega_M = 0.5\pi$, and a copy every 2π.

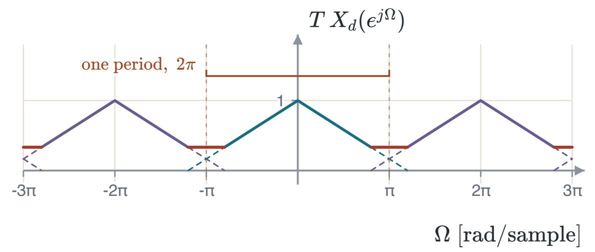

FIGURE 7.48 The same signal with $T = 0.6$ s: $\Omega_M = 1.2\pi > \pi$, and neighbouring copies overlap, red.

The spectrum of the samples

The spectrum of the sequence $x_d[n]$ follows from the sampled spectrum of Section 7.1 by a change of variable. Step 1 writes both transforms as sums over the same samples. The impulse $\delta(t - nT)$ has transform $\int \delta(t - nT)e^{-j\omega t}\,\mathrm{d}t = e^{-j\omega nT}$ by sifting, so the transform of $x_p(t) = \sum_n x_c(nT)\,\delta(t - nT)$ is a sum of such terms. The second line is the definition of the transform of $x_d[n] = x_c(nT)$.

$$X_p(j\omega) = \sum_{n=-\infty}^{\infty} x_c(nT)\, e^{-j\omega nT}$$
$$X_d(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x_c(nT)\, e^{-j\Omega n}$$

Step 2 compares them. The two sums are the same when $\omega T = \Omega$, that is $\omega = \Omega/T$. Then substitute the sampled spectrum and use $k\omega_s T = 2\pi k$:

THE SPECTRUM OF THE SEQUENCE OF SAMPLES

$$X_d(e^{j\Omega}) = X_p\Big(j\frac{\Omega}{T}\Big) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\Big(j\Big(\frac{\Omega}{T} - k\omega_s\Big)\Big)$$
$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\Big(j\,\frac{\Omega - 2\pi k}{T}\Big)$$

The second line writes $\frac{\Omega}{T} - k\omega_s = \frac{\Omega - k\omega_s T}{T} = \frac{\Omega - 2\pi k}{T}$. The copy k sits at $\Omega = 2\pi k$ for every T; changing T only stretches the drawing, because the time axis $t = nT$ has become the integer n. The result repeats every 2π in Ω, as every discrete-time spectrum must.

The equivalent continuous-time system

Follow a band-limited input with $X_c(j\omega) = 0$ for $|\omega| \geq \omega_s/2$ through the chain. The D/C converter forms $\sum_n y_d[n]\,\delta(t - nT)$, whose transform is $\sum_n y_d[n]e^{-j\omega nT} = Y_d(e^{j\omega T})$ by Step 1, and multiplies by the filter of gain T on $|\omega| < \omega_s/2$. Inside that band:

$$Y_c(j\omega) = T\,Y_d(e^{j\omega T}) = T\,H_d(e^{j\omega T})\,X_d(e^{j\omega T})$$
$$= T\,H_d(e^{j\omega T})\cdot\frac{1}{T}\,X_c(j\omega) = H_d(e^{j\omega T})\,X_c(j\omega).$$

The second equality is the convolution property of the discrete-time system, $Y_d = H_d X_d$. The third uses the spectrum of the samples at $\Omega = \omega T$: for $|\omega| < \omega_s/2$ only the copy $k = 0$ lies there, and it is $X_c(j\omega)/T$. Outside the band the D/C filter gives zero.

KEY RESULT: THE EQUIVALENT SYSTEM

$$H_{\text{eff}}(j\omega) = \begin{cases} H_d\big(e^{j\omega T}\big), & |\omega| < \omega_s/2 \\ 0, & |\omega| > \omega_s/2 \end{cases}$$

This holds for every input with $X_c(j\omega) = 0$ at $|\omega| \geq \omega_s/2$. For other inputs the chain is not even time-invariant: a short pulse may fall between two samples and vanish, while the same pulse shifted onto a sample instant does not. To design H_d for a wanted H_{eff}, read the key result backwards: $H_d(e^{j\Omega}) = H_{\text{eff}}(j\Omega/T)$ for $|\Omega| < \pi$, repeated every 2π.

An example: the three-point average $y_d[n] = \frac{1}{4}x_d[n+1] + \frac{1}{2}x_d[n] + \frac{1}{4}x_d[n-1]$. A shift by one sample multiplies the transform by $e^{\pm j\Omega}$, so

$$H_d(e^{j\Omega}) = \frac{1}{4}e^{j\Omega} + \frac{1}{2} + \frac{1}{4}e^{-j\Omega} = \frac{1}{2} + \frac{1}{4}\big(e^{j\Omega} + e^{-j\Omega}\big) = \frac{1}{2} + \frac{1}{2}\cos\Omega = \frac{1}{2}(1 + \cos\Omega),$$

using $e^{j\Omega} + e^{-j\Omega} = 2\cos\Omega$. With $T = 0.25$ s, $\omega_s/2 = \pi/T = 4\pi$ rad/s and $H_{\text{eff}}(j\omega) = \frac{1}{2}\big(1 + \cos(0.25\,\omega)\big)$ for $|\omega| < 4\pi$. At $\omega = 2\pi$ it is $\frac{1}{2}(1 + \cos(\pi/2)) = \frac{1}{2}$. At $\omega = 6\pi > 4\pi$ it is 0: the D/C converter removes it, although H_d itself repeats there.

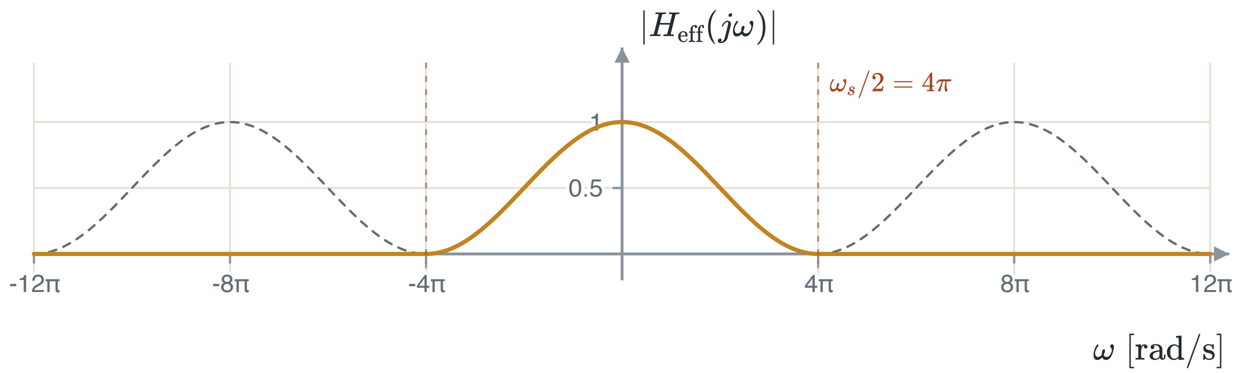

FIGURE 7.49 The three-point average with $T = 0.25$ s as a continuous-time system: $H_{\text{eff}}(j\omega) = \frac{1}{2}\big(1 + \cos(\omega T)\big)$ for $|\omega| < \omega_s/2 = 4\pi$ rad/s, amber, and zero beyond. The grey dashed curve is the periodic $H_d(e^{j\omega T})$ that the D/C converter cuts off.

✎ EXAMPLE 7.13 — A DIGITAL CUTOFF IN HERTZ

GIVEN A digital low-pass that keeps $|\Omega| < \Omega_c = \pi/4$, run at $f_s = 8$ kHz and then at $f_s = 44.1$ kHz.

FIND Its cutoff in rad/s and in hertz at each rate.

METHOD Read the key result backwards: the filter keeps $|\omega T| < \Omega_c$, so $\omega_c = \Omega_c/T = \Omega_c f_s$ and $f_c = \omega_c/2\pi = \Omega_c f_s/2\pi$.

SOLUTION At $f_s = 8$ kHz:

$$\omega_c = \frac{\pi}{4} \cdot 8000 = 2000\pi \text{ rad/s}, \qquad f_c = \frac{2000\pi}{2\pi} = 1000 \text{ Hz} = 1 \text{ kHz}.$$

At $f_s = 44.1$ kHz: $f_c = \frac{\pi/4}{2\pi} \cdot 44.1 = \frac{44.1}{8} = 5.5125$ kHz. For $\Omega_c = \pi/4$ in general, $f_c = f_s/8$.

CHECK The cutoff must lie below $f_s/2$, because $\Omega_c < \pi$: $1 < 4$ kHz and $5.5125 < 22.05$ kHz. The filter stores only Ω_c; the sampling rate decides where it acts. A sound card that changes its rate moves every cutoff with it, so a digital filter is designed for the rate it runs at. For a filter that keeps $|\Omega| < \pi/2$ at 16 kHz, $f_c = \frac{\pi/2}{2\pi} \cdot 16 = 4$ kHz.

A digital differentiator

The derivative $\mathrm{d}x_c/\mathrm{d}t$ has transform $j\omega X_c(j\omega)$. So a band-limited differentiator is

$$H_{\text{eff}}(j\omega) = \begin{cases} j\omega, & |\omega| < \omega_s/2 \\ 0, & |\omega| > \omega_s/2 \end{cases} \implies H_d(e^{j\Omega}) = H_{\text{eff}}\Big(j\frac{\Omega}{T}\Big) = j\,\frac{\Omega}{T}, \quad |\Omega| < \pi.$$

Outside $|\Omega| < \pi$ the response repeats every 2π. Its magnitude is the ramp $|\Omega|/T$, which reaches π/T at the band edge; with $T = 0.25$ s that is 4π, and at $\Omega = \pi/2$ it is $(\pi/2)/0.25 = 2\pi$. Its phase is $+\pi/2$ for $\Omega > 0$ and $-\pi/2$ for $\Omega < 0$, and at $\Omega = \pi$ the response jumps from $j\pi/T$ to $-j\pi/T$.

✎ EXAMPLE 7.14 — THE DIFFERENTIATOR ON A TONE

GIVEN The digital differentiator with $f_s = 1$ kHz, and the input tone $x_c(t) = \cos(2\pi f_0 t)$.

FIND $y_c(t)$ for $f_0 = 100$ Hz, and its peak for $f_0 = 700$ Hz.

METHOD Write the sampled tone as two complex exponentials, multiply each by H_d at its own frequency, and convert back with $\Omega_0 = \omega_0 T$.

SOLUTION The samples are $x_d[n] = \cos(\Omega_0 n) = \frac{1}{2}e^{j\Omega_0 n} + \frac{1}{2}e^{-j\Omega_0 n}$ with $\Omega_0 = \omega_0 T$. For $0 < \Omega_0 < \pi$, $H_d(e^{j\Omega_0}) = j\Omega_0/T$ and $H_d(e^{-j\Omega_0}) = -j\Omega_0/T$, so

$$y_d[n] = \frac{1}{2}\cdot\frac{j\Omega_0}{T}e^{j\Omega_0 n} - \frac{1}{2}\cdot\frac{j\Omega_0}{T}e^{-j\Omega_0 n} = \frac{\Omega_0}{T}\cdot j\cdot\frac{e^{j\Omega_0 n} - e^{-j\Omega_0 n}}{2}$$
$$= \frac{\Omega_0}{T}\cdot j\cdot j\sin(\Omega_0 n) = -\frac{\Omega_0}{T}\sin(\Omega_0 n) = -\omega_0\sin(\omega_0 nT).$$

The gain is $\Omega_0/T = \omega_0$ and the phase shift is $\pi/2$, since $-\sin\theta = \cos(\theta + \pi/2)$. The D/C converter returns $y_c(t) = -\omega_0\sin(\omega_0 t) = \mathrm{d}x_c/\mathrm{d}t$. At $f_0 = 100$ Hz, $\Omega_0 = 2\pi\cdot 100\cdot 0.001 = 0.2\pi < \pi$, and the output peaks at $\omega_0 = 200\pi \approx 628.3$ per second, that is 0.628 per ms. At $f_0 = 700$ Hz, $\Omega_0 = 1.4\pi > \pi$: the tone lies above $f_s/2 = 500$ Hz, so it folds first, to $f_s - f_0 = 300$ Hz, because $\cos(1.4\pi n) = \cos(2\pi n - 0.6\pi n) = \cos(0.6\pi n)$. The filter then differentiates the 300 Hz alias, and the output peaks at $2\pi\cdot 300 = 600\pi \approx 1885$ per second.

CHECK For $f_0 = 100$ Hz the output is the true derivative of the input. For $f_0 = 700$ Hz it is not: the true derivative would peak at $2\pi\cdot 700$ per second at 700 Hz, while the chain gives a 300 Hz tone with a smaller peak. The equivalent system holds only for band-limited inputs.

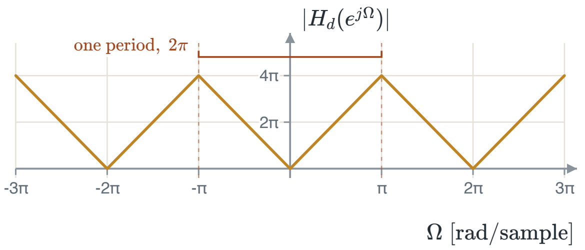

FIGURE 7.50 The magnitude of the digital differentiator for $T = 0.25$ s: a ramp to $\pi/T = 4\pi$ at $\Omega = \pm\pi$, repeated every 2π.

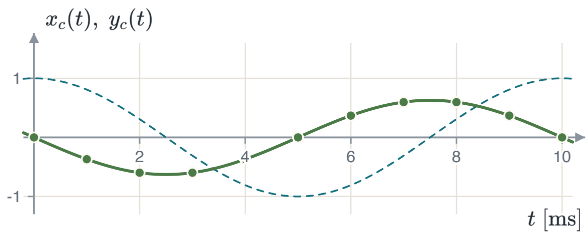

FIGURE 7.51 A 100 Hz tone, dashed, and the differentiator output at $f_s = 1$ kHz, in units of per millisecond: $-0.2\pi\sin(0.2\pi t)$ with t in ms, peak 0.628.

A half-sample delay

A delay by Δ multiplies the transform by $e^{-j\omega\Delta}$. A band-limited delay by half a sampling period, $\Delta = T/2$, is therefore

$$H_{\text{eff}}(j\omega) = e^{-j\omega T/2},\ |\omega| < \omega_s/2 \quad\Longrightarrow\quad H_d(e^{j\Omega}) = e^{-j\frac{\Omega}{T}\cdot\frac{T}{2}} = e^{-j\Omega/2},\ |\Omega| < \pi.$$

Its gain is $|H_d| = 1$, and its phase $-\Omega/2$ is a straight line: every frequency is delayed by half a sample. The output is $y_d[n] = y_c(nT) = x_c(nT - T/2)$, the curve through the samples read half a step earlier. There is no formula $x_d[n - \frac{1}{2}]$: a sequence has no value at $n - \frac{1}{2}$. For example, with $T = 1$ ms and $x_c(t) = \cos(2\pi\cdot 250\,t)$, t in seconds, $y_d[0] = x_c(-0.0005) = \cos(-2\pi\cdot 250\cdot 0.0005) = \cos(-\pi/4) \approx 0.707$.

The impulse response is the inverse transform of H_d over one period:

THE IMPULSE RESPONSE OF THE HALF-SAMPLE DELAY

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\Omega/2}\, e^{j\Omega n}\, \mathrm{d}\Omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\Omega(n-\frac{1}{2})}\, \mathrm{d}\Omega = \frac{1}{2\pi}\left[\frac{e^{j\Omega(n-\frac{1}{2})}}{j(n-\frac{1}{2})}\right]_{-\pi}^{\pi}$$
$$= \frac{1}{2\pi}\cdot\frac{2j\sin\big(\pi(n-\frac{1}{2})\big)}{j(n-\frac{1}{2})} = \frac{\sin\big(\pi(n-\frac{1}{2})\big)}{\pi(n-\frac{1}{2})} = \operatorname{sinc}\Big(\pi\big(n-\tfrac{1}{2}\big)\Big)$$

The exponents combine as $-\Omega/2 + \Omega n = \Omega(n-\frac{1}{2})$, and $n-\frac{1}{2}$ is never zero, so the antiderivative has no zero denominator. Evaluating at $\pm\pi$ uses $e^{j\theta} - e^{-j\theta} = 2j\sin\theta$. Here $\operatorname{sinc}(\theta) = \sin\theta/\theta$, unnormalised. Since $\sin\big(\pi(n-\frac{1}{2})\big) = (-1)^{n+1}$, $h[n] = (-1)^{n+1}/\big(\pi(n-\frac{1}{2})\big)$: it is never zero, and it decays only like $1/n$, so every input sample contributes to every output sample. The two central values are equal: $h[0] = \frac{-1}{-\pi/2} = \frac{2}{\pi}$ and $h[1] = \frac{1}{\pi/2} = \frac{2}{\pi}$.

A second route gives the same $h[n]$. Feed the chain $x_c(t) = \sin(\pi t/T)/(\pi t)$, whose transform is 1 on $|\omega| < \pi/T$ (Example 7.4 with $W = \pi/T$). Its samples are $x_d[n] = \sin(\pi n)/(\pi n T) = 0$ for $n \neq 0$ and $x_d[0] = 1/T$, so $x_d[n] = \frac{1}{T}\delta[n]$ and the output is $\frac{1}{T}h[n]$. The output is also $x_c(nT - T/2)$, so $h[n] = T\,x_c(nT - \frac{T}{2}) = \frac{\sin(\pi(n-\frac{1}{2}))}{\pi(n-\frac{1}{2})}$.

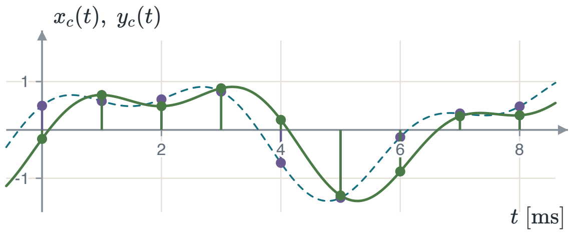

FIGURE 7.52 $x_c(t) = \sin(2\pi\, 0.15t) + 0.5\cos(2\pi\, 0.32t)$ with t in ms, sampled at $T = 1$ ms, violet. The half-sample delay gives $y_d[n] = x_c(nT - T/2)$, green, the samples of the delayed curve.

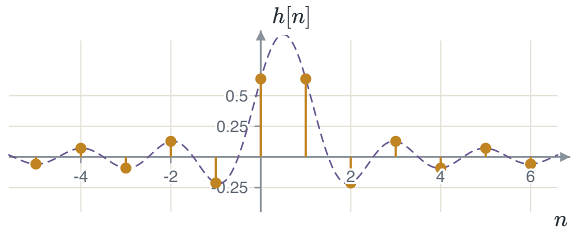

FIGURE 7.53 The impulse response $h[n] = \operatorname{sinc}\big(\pi(n-\frac{1}{2})\big)$, with $\operatorname{sinc}(\theta) = \sin\theta/\theta$: the kernel shifted by half a sample, dashed, read at the integers. Every n carries a value.

Quantization

A real C/D converter with B bits also rounds each sample to one of 2^B levels, so that the result $x_q[n]$ is a number the computer can store. Over the range -1 to 1 the levels are a step Δ apart:

$$\Delta = \frac{2}{2^B}, \qquad e[n] = x_q[n] - x[n], \qquad |e[n]| \le \frac{\Delta}{2}.$$

Each value goes to the level of its own step, and the level sits in the middle of the step, so no value is more than half a step from its level. One more bit halves Δ, and with it the largest error. With $B = 4$, $\Delta = 2/16 = 1/8$ and $|e| \le 1/16$. Rounding is a second loss, besides aliasing: it cannot be undone, and no sampling rate removes it. Only more bits make it smaller.

The size of the error is measured by its power. Take the error as spread evenly over $-\Delta/2 < e < \Delta/2$, with density $1/\Delta$. Its mean square is

$$\overline{e^2} = \frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} e^2\,\mathrm{d}e = \frac{1}{\Delta}\left[\frac{e^3}{3}\right]_{-\Delta/2}^{\Delta/2} = \frac{1}{\Delta}\cdot\frac{1}{3}\left(\frac{\Delta^3}{8} + \frac{\Delta^3}{8}\right) = \frac{\Delta^2}{12}.$$

A full-scale sine $\sin(\omega_0 t)$ has power $\frac{1}{2}$, the mean of $\sin^2$. With $\Delta^2 = 4/2^{2B}$ the signal-to-noise ratio in decibels is

ABOUT SIX DECIBELS PER BIT

$$\begin{aligned}\mathrm{SNR} &= 10\log_{10}\frac{1/2}{\Delta^2/12} = 10\log_{10}\frac{6}{\Delta^2} = 10\log_{10}\frac{6\cdot 2^{2B}}{4} = 10\log_{10}\left(1.5\cdot 2^{2B}\right)\\ &= 10\log_{10}1.5 + 2B\cdot 10\log_{10}2 \approx 1.76 + 6.02\,B\ \mathrm{dB}\end{aligned}$$

The last line uses $\log(ab) = \log a + \log b$ and $\log(2^{2B}) = 2B\log 2$, with $10\log_{10}1.5 \approx 1.76$ and $20\log_{10}2 \approx 6.02$. Each extra bit adds about 6 dB, so going from 12 to 14 bits adds about $2\times 6.02 \approx 12$ dB. The rule gives 19.8 dB at 3 bits, 49.9 dB at 8 and 98.1 dB at 16, the CD format; the ratio measured on a sampled full-scale sine lies within 1 dB of the rule from 3 bits up.

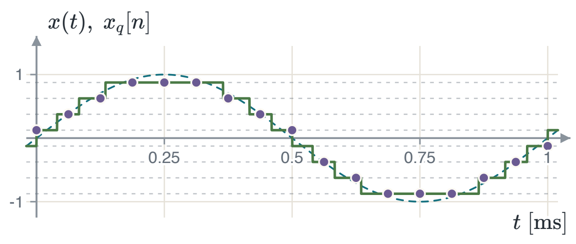

FIGURE 7.54 A full-scale 1 kHz sine, t in ms, sampled 16 times a cycle and rounded with $B = 3$ bits: 8 levels, $\Delta = 0.25$, dotted lines.

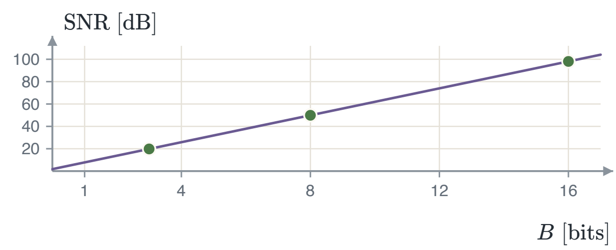

FIGURE 7.55 The rule $\mathrm{SNR} \approx 6.02B + 1.76$ dB, with the three cases $B = 3$, 8 and 16 marked.

7.6 Decimation and interpolation

A sequence can be sampled too: keep every N-th value and set the others to zero. This section derives what that does to the spectrum, and then uses it to lower the rate of a sequence (**decimation**) and to raise it (**interpolation**). Every spectrum here is a discrete-time transform, so the frequency variable is ω in rad/sample, as in Chapter 6, and every spectrum repeats every 2π.

The running sequence of this section is $x[n] = \left(\sin(\pi n/8)/(\pi n/8)\right)^2$, with $x[0] = 1$. Write it as $x[n] = 64\,g[n]^2$ with $g[n] = \sin(Wn)/(\pi n)$ and $W = \pi/8$, because $\sin(\pi n/8)/(\pi n/8) = 8\sin(Wn)/(\pi n)$. The transform of g is 1 on $|\omega| \le W$ and 0 on $W < |\omega| \le \pi$ (Chapter 6). Squaring in time is a periodic convolution divided by 2π, and the convolution of that rectangle with itself is the triangle $2W - |\omega|$ on $|\omega| \le 2W$, as in Example 7.5. So

$$X(e^{j\omega}) = 64\cdot\frac{1}{2\pi}(2W - |\omega|) = \frac{32}{\pi}\left(\frac{\pi}{4} - |\omega|\right), \qquad |\omega| \le \frac{\pi}{4},$$

a triangle of peak $\frac{32}{\pi}\cdot\frac{\pi}{4} = 8$ that reaches zero at $\omega_M = \pi/4$, repeated every 2π.

Sampling a sequence

SAMPLING A SEQUENCE

$$p[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN], \qquad x_p[n] = x[n]\, p[n]$$

$$x_p[n] = \sum_{k=-\infty}^{\infty} x[n]\, \delta[n - kN] = \sum_{k=-\infty}^{\infty} x[kN]\, \delta[n - kN] = \begin{cases} x[n], & n = kN \\ 0, & \text{otherwise} \end{cases}$$

The sampler multiplies $x[n]$ by a unit sample every N samples; the integer N is the sampling period. The second step uses $x[n]\,\delta[n - kN] = x[kN]\,\delta[n - kN]$, because $\delta[n - kN]$ is zero except at $n = kN$. For example, with $x[n] = (0.5)^{|n|}$ and $N = 3$, $x_p[4] = 0$, because 4 is not a multiple of 3.

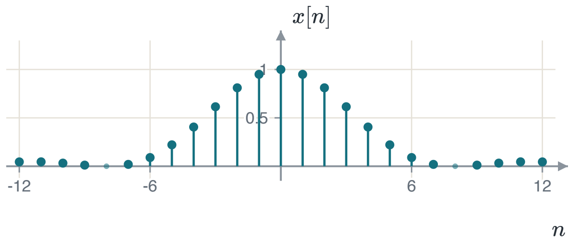

FIGURE 7.56 The running sequence $x[n] = \left(\sin(\pi n/8)/(\pi n/8)\right)^2$.

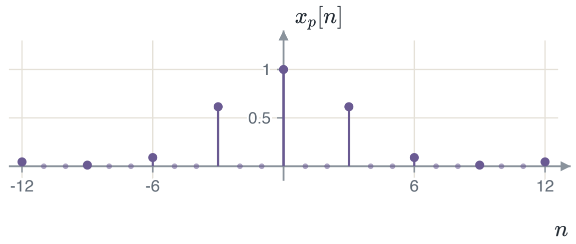

FIGURE 7.57 The sampled sequence $x_p[n]$ with $N = 3$: every third value is kept and the others are zero.

The spectrum of a sampled sequence

The derivation follows the continuous-time one of Section 7.1. Step 1 finds the transform of $p[n]$. It is periodic with period N, so it has a discrete-time Fourier series with coefficients

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} p[n]\, e^{-jk(2\pi/N)n} = \frac{1}{N}\left(1 \cdot e^0\right) = \frac{1}{N},$$

since in one period $0 \le n \le N - 1$ only $p[0] = 1$ is non-zero. A periodic sequence has impulses of weight $2\pi a_k$ at $\omega = 2\pi k/N$ (Chapter 6), so

STEP 1: THE SAMPLING SEQUENCE

$$P(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi \cdot \frac{1}{N}\, \delta\!\left(\omega - \frac{2\pi k}{N}\right) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s), \qquad \omega_s = \frac{2\pi}{N}$$

One period of 2π holds the N impulses $k = 0, 1, \ldots, N - 1$, each of weight $2\pi/N$; with $N = 4$ that is four impulses of weight $\pi/2$. Here $\omega_s = 2\pi/N$ is the sampling frequency of the sequence, in rad/sample.

Step 2 is the multiplication property of Chapter 6: a product in time is a periodic convolution over one period, divided by 2π. Step 3 substitutes P and sifts. Over $0 \le \theta < 2\pi$ the impulses of P are those with $k = 0, \ldots, N - 1$:

$$X_p(e^{j\omega}) = \frac{1}{2\pi}\int_0^{2\pi} P(e^{j\theta})\, X\left(e^{j(\omega-\theta)}\right) \mathrm{d}\theta = \frac{1}{2\pi}\cdot\frac{2\pi}{N}\sum_{k=0}^{N-1}\int_0^{2\pi}\delta(\theta-k\omega_s)\, X\left(e^{j(\omega-\theta)}\right)\mathrm{d}\theta$$
$$= \frac{1}{N}\sum_{k=0}^{N-1} X\left(e^{j(\omega-k\omega_s)}\right).$$

KEY RESULT: THE SPECTRUM OF A SAMPLED SEQUENCE

$$X_p(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1} X\left(e^{j(\omega-2\pi k/N)}\right)$$

N copies of $X(e^{j\omega})$ in each period, spaced $2\pi/N$ apart and scaled by $1/N$. Each copy reaches ω_M on either side of its centre, so neighbours stay apart when $2\pi/N > 2\omega_M$, that is

$$\omega_s > 2\omega_M \iff \frac{2\pi}{N} > 2\omega_M \iff \omega_M < \frac{\pi}{N}.$$

For the running sequence, $\omega_M = \pi/4$: with $N = 3$, $\pi/4 < \pi/3$ and the copies stand apart; with $N = 4$, $\pi/4 = \pi/4$ and they just touch; from $N = 5$ on they overlap. A sequence with $X(e^{j\omega}) = 0$ for $2\pi/7 \le |\omega| \le \pi$ needs $2\pi/7 < \pi/N$, that is $N < 3.5$, so the largest N with no aliasing is 3.

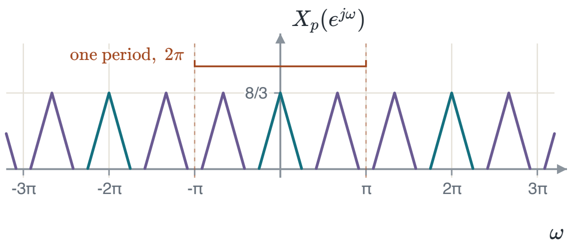

FIGURE 7.58 The sampled spectrum of the running sequence with $N = 3$: copies of height $8/3$, spaced $2\pi/3$ apart, that stand clear.

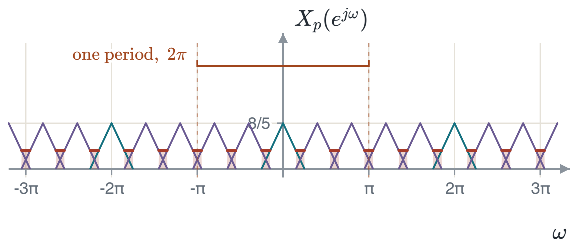

FIGURE 7.59 The same with $N = 5$: the copies, $2\pi/5$ apart, overlap because $\pi/4 > \pi/5$. Their sum is red.

Recovering the sequence

Without aliasing, an ideal discrete-time low-pass filter recovers $x[n]$ from $x_p[n]$. It keeps the copy at $\omega = 0$ and removes the others, and its gain N undoes the factor $1/N$:

$$H(e^{j\omega}) = \begin{cases} N, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}, \qquad \omega_M < \omega_c < \omega_s - \omega_M, \qquad X_r(e^{j\omega}) = N\cdot\frac{1}{N}X(e^{j\omega}) = X(e^{j\omega}).$$

Like every discrete-time frequency response, H repeats every 2π. With no aliasing, $\omega_c = \pi/N = \omega_s/2$ always lies in the range. For the running sequence with $N = 2$ the range is $\pi/4 < \omega_c < \pi - \pi/4 = 3\pi/4$, so $\omega_c = \pi/2$ recovers it.

Decimation

The zeros of $x_p[n]$ carry nothing, so storing them wastes memory. **Decimation** by N keeps every N-th value and closes the gaps:

$$x_b[n] = x_p[nN] = x[nN].$$

For example, $x[n] = n$ for $0 \le n \le 11$ and $N = 4$ give $x_b[2] = x[2 \cdot 4] = x[8] = 8$. If $x[n]$ holds samples of $x(t)$ taken every T, then $x_b[n]$ holds samples taken every NT: the sampling rate falls by the factor N, which is why decimation is also called **downsampling**. Its spectrum follows from the definition and one change of index.

$$\begin{aligned} X_b(e^{j\omega}) &= \sum_{k=-\infty}^{\infty} x_b[k]\, e^{-j\omega k} = \sum_{k=-\infty}^{\infty} x_p[kN]\, e^{-j(\omega/N)kN} \\ &= \sum_{n=-\infty}^{\infty} x_p[n]\, e^{-j(\omega/N)n} = X_p\big(e^{j\omega/N}\big). \end{aligned}$$

The first line puts $x_b[k] = x_p[kN]$ into the definition and writes $\omega k = (\omega/N)(kN)$. The second line sets $n = kN$; the terms of $x_p[n]$ with n not a multiple of N are zero, so adding them changes nothing, and the sum is over all n.

DECIMATION STRETCHES THE SPECTRUM

$$X_b(e^{j\omega}) = X_p\big(e^{j\omega/N}\big) = \frac{1}{N}\sum_{k=0}^{N-1} X\big(e^{j(\omega-2\pi k)/N}\big)$$

The frequency axis of X_p is stretched by N. A band edge ω_M moves to $N\omega_M$, and the copies at $2\pi k/N$ move out to $2\pi k$, so the period stays 2π. For the running sequence with $N = 3$ the band widens from $\pi/4$ to $3\pi/4$; with $N = 2$ it widens to $\pi/2$.

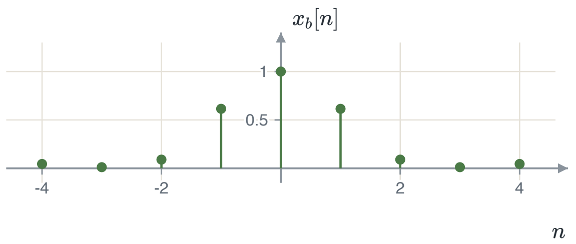

FIGURE 7.60 The running sequence decimated by $N = 3$: $x_b[n] = x[3n]$.

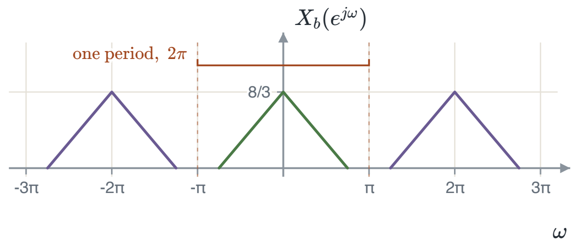

FIGURE 7.61 Its spectrum: the triangle of the running sequence stretched by 3, now reaching $3\pi/4$, with its copies at every multiple of 2π.

Filter before you decimate

Decimation stretches the band edge to $N\omega_M$. If $N\omega_M > \pi$, the stretched copies overlap, and that is aliasing. So a sequence is low-pass filtered to $|\omega| < \pi/N$ before it is decimated:

$$x[n] \to \boxed{H_d(e^{j\omega})} \to \boxed{\downarrow N} \to x_b[n], \qquad H_d(e^{j\omega}) = \begin{cases} 1, & |\omega| < \pi/N \\ 0, & \pi/N < |\omega| \le \pi. \end{cases}$$

After the filter the band edge is at most π/N, so the stretch by N keeps it inside $|\omega| \le \pi$. The filter comes first; after the decimator no filter can separate a folded tone from wanted content.

EXAMPLE 7.15 — DECIMATING TWO TONES

GIVEN Tones of 500 Hz and 3 kHz sampled at 8 kHz: $x[n] = \cos(\pi n/8) + \cos(3\pi n/4)$.

FIND What decimation by $N = 2$ gives with and without the prefilter, and which tone keeps its pitch when $N = 4$ with no filter.

METHOD Map each tone to rad/sample with $\omega = 2\pi f/f_s$, multiply by N, and bring the result into $-\pi < \omega \le \pi$ by adding a multiple of 2π. Convert back to hertz at the new rate f_s/N.

SOLUTION The tones sit at $\omega_1 = 2\pi \cdot 500/8000 = \pi/8$ and $\omega_2 = 2\pi \cdot 3000/8000 = 3\pi/4$. With $N = 2$ and no filter they move to $2 \cdot \pi/8 = \pi/4$ and $2 \cdot 3\pi/4 = 3\pi/2$. The second is outside $|\omega| \le \pi$; subtract 2π: $3\pi/2 - 2\pi = -\pi/2$. At the new rate $8/2 = 4$ kHz, $\pi/4$ is $\frac{\pi/4}{2\pi} \cdot 4000 = 500$ Hz and $\pm\pi/2$ is $\frac{\pi/2}{2\pi} \cdot 4000 = 1000$ Hz. So the 3 kHz tone is heard at 1 kHz. With the prefilter, $|\omega| < \pi/2$ keeps $\pi/8$ and removes $3\pi/4 > \pi/2$, and only the 500 Hz tone remains. With $N = 4$ and no filter the new rate is 2 kHz: $4 \cdot \pi/8 = \pi/2 < \pi$, so 500 Hz keeps its pitch, while $4 \cdot 3\pi/4 = 3\pi$, which is $3\pi - 2\pi = \pi$, that is 1 kHz.

CHECK In hertz, the fold rule of Section 7.4 at the new rate gives the same answers: at 4 kHz, $|3 - 4| = 1$ kHz; at 2 kHz, $|3 - 2| = 1$ kHz, while $500 < 1000$ Hz $= f_s/2$ stays where it is.

Interpolation

Interpolation by N raises the rate. Step 1 inserts $N - 1$ zeros between neighbouring samples; with $N = 4$ that is three zeros in each gap:

$$x_{(N)}[n] = \begin{cases} x_b[n/N], & n = 0, \pm N, \pm 2N, \dots \\ 0, & \text{otherwise.} \end{cases}$$

This is the time expansion of Chapter 6. Only the terms $n = kN$ are non-zero, so its transform is

$$X_{(N)}(e^{j\omega}) = \sum_n x_{(N)}[n]\, e^{-j\omega n} = \sum_k x_b[k]\, e^{-j\omega kN} = \sum_k x_b[k]\, e^{-j(N\omega)k} = X_b\big(e^{jN\omega}\big).$$

The axis is compressed by N, so the spectrum repeats every $2\pi/N$. One period of 2π now holds N copies: the wanted one at $\omega = 0$ and $N - 1$ **images** centred at $2\pi k/N$, $k = 1, \dots, N-1$. Step 2 removes the images with a low-pass filter:

$$y[n] = \big(x_{(N)} * h\big)[n], \qquad H(e^{j\omega}) = \begin{cases} N, & |\omega| < \pi/N \\ 0, & \pi/N < |\omega| \le \pi. \end{cases}$$

The gain N keeps the old samples at their values. The impulse response of the filter is

$$\begin{aligned} h[n] &= \frac{1}{2\pi} \int_{-\pi/N}^{\pi/N} N\, e^{j\omega n}\, \mathrm{d}\omega = \frac{N}{2\pi} \left[\frac{e^{j\omega n}}{jn} \right]_{-\pi/N}^{\pi/N} = \frac{N}{2\pi} \cdot \frac{2j \sin(\pi n/N)}{jn} \\ &= \frac{\sin(\pi n/N)}{\pi n/N} = \operatorname{sinc}\Big(\frac{\pi n}{N}\Big), \qquad \operatorname{sinc}(\theta) = \frac{\sin\theta}{\theta}, \end{aligned}$$

with $h[0] = \frac{1}{2\pi} \cdot N \cdot \frac{2\pi}{N} = 1$ from the integral directly. At the other multiples of N, $h[mN] = \sin(\pi m)/(\pi m) = 0$. So at $n = kN$ only the term $i = k$ of the convolution survives: $y[kN] = \sum_i x_b[i]\, h[(k-i)N] = x_b[k]$. The kept samples do not move; the filter only fills the zeros.

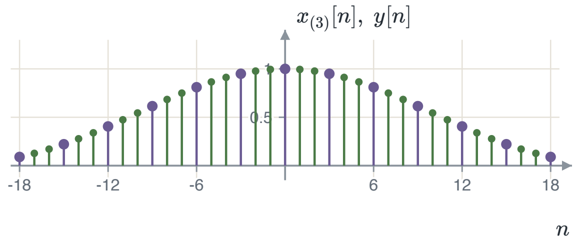

FIGURE 7.62 Interpolation by $N = 3$ of the running sequence: the kept samples, violet, at $n = 3k$, and the values the low-pass puts into the zeros, green.

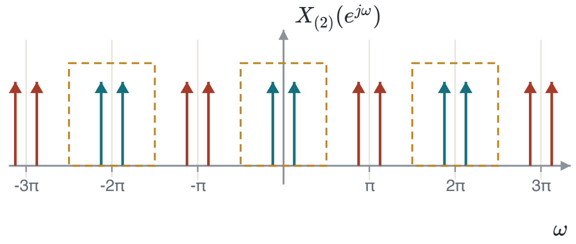

FIGURE 7.63 $x_b[n] = \cos(\pi n/4)$ with one zero inserted after every sample, $N = 2$: the wanted lines at $\pm\pi/8$, cyan, and the images at $\pm 7\pi/8$, red, in every period. The low-pass, dashed, keeps $|\omega| < \pi/2$.

EXAMPLE 7.16 — THE IMAGES OF A TONE

GIVEN A 500 Hz tone sampled at 4 kHz, $x_b[n] = \cos(\pi n/4)$, interpolated by $N = 2$ to 8 kHz.

FIND Where the images lie, what they sound like without the filter, and how many copies a period holds when $N = 4$.

METHOD Use $X_{(N)}(e^{j\omega}) = X_b(e^{jN\omega})$: a line of X_b at $\omega_0 + 2\pi m$ moves to $(\omega_0 + 2\pi m)/N$.

SOLUTION X_b has lines at $\pm\pi/4 + 2\pi m$. With $N = 2$ they move to $(\pm\pi/4 + 2\pi m)/2 = \pm\pi/8 + \pi m$. For even m these are $\pm\pi/8$ plus multiples of 2π, the wanted tone: at 8 kHz, $\frac{\pi/8}{2\pi} \cdot 8000 = 500$ Hz. For odd m they are the images; in $-\pi < \omega \le \pi$, $m = 1$ gives $-\pi/8 + \pi = 7\pi/8$ and $m = -1$ gives $\pi/8 - \pi = -7\pi/8$. At 8 kHz, $\frac{7\pi/8}{2\pi} \cdot 8000 = 3500$ Hz, so without the filter the images sound as a 3.5 kHz whistle. The low-pass of gain 2 and cutoff $\pi/2$ removes them. With $N = 4$ the spectrum repeats every $2\pi/4$, so one period of 2π holds the wanted copy and 3 images: 4 copies in all.

CHECK $7\pi/8 + \pi/8 = \pi$: each image is the mirror of the tone about $\pi/2$, the new $f_s/4$. In hertz, $3500 + 500 = 4000$ Hz, the old sampling rate, as for the copies of a sampled tone at $f_s \pm f_0$.

A rate change by L/M

To change the rate by a rational factor L/M, go up by L, filter once, and go down by M:

$$x[n] \to \boxed{\uparrow L} \to \boxed{H,\ \text{gain } L,\ \omega_c = \min\left(\tfrac{\pi}{L}, \tfrac{\pi}{M}\right)} \to \boxed{\downarrow M} \to y[m].$$

One filter does both jobs. The interpolator needs a cutoff of π/L to remove the images of $\uparrow L$, and the decimator needs a cutoff of π/M to prevent the aliasing of $\downarrow M$. Both run at the same high rate, so one low-pass with the smaller of the two cutoffs meets both conditions.

EXAMPLE 7.17 — FROM 48 KHZ TO 44.1 KHZ

GIVEN A recording at 48 kHz that must be converted to 44.1 kHz.

FIND L, M, the rate at which the filter runs, and its cutoff.

METHOD Write the ratio of the rates in lowest terms, with the greatest common divisor from the prime factorisations.

SOLUTION $44100 = 2^2 \cdot 3^2 \cdot 5^2 \cdot 7^2$ and $48000 = 2^7 \cdot 3 \cdot 5^3$. The common factors are $2^2 \cdot 3 \cdot 5^2 = 300$, so

$$\frac{44100}{48000} = \frac{44100/300}{48000/300} = \frac{147}{160} \quad \Longrightarrow \quad L = 147,\ M = 160.$$

The filter runs at $48 \cdot 147 = 7056$ kHz, with gain 147 and cutoff $\min(\pi/147, \pi/160) = \pi/160$.

CHECK $48 \cdot \frac{147}{160} = \frac{7056}{160} = 44.1$ kHz. The reverse conversion, from 44.1 kHz to 48 kHz, raises the rate by $48000/44100 = 160/147$: up by 160, down by 147.

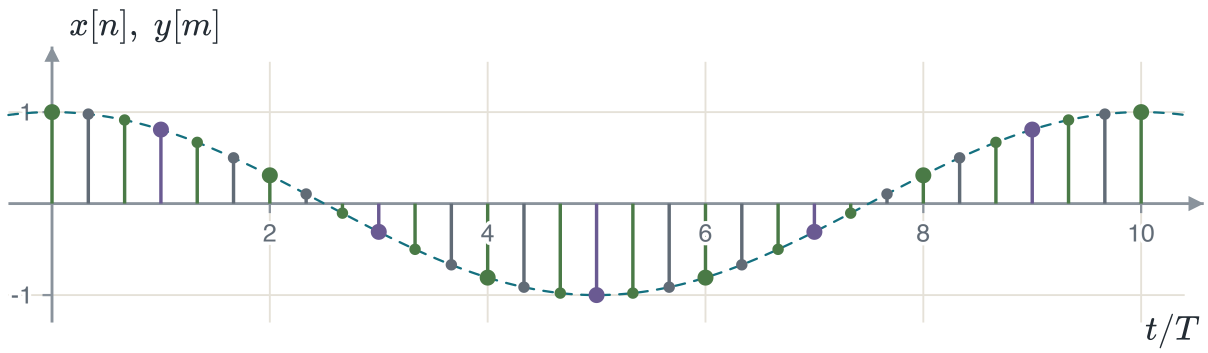

FIGURE 7.64 A rate change by $L/M = 3/2$ for $x[n] = \cos(\pi n/5)$, time in units of the old spacing T. Violet: the old samples. Up by 3 and the filter put values between them; down by 2 keeps every second one. The output $y[m]$, green, has 3 samples in every 2 units of time.

7.7 Summary

The three tables collect the results of the chapter. The first two are in continuous time, with ω in rad/s; the third mixes the two, with Ω or ω in rad/sample for sequences as in Sections 7.5 and 7.6.

TABLE 7.1 Sampling and the sampling theorem.

RESULT	STATEMENT
Sampler	$x_p(t) = x(t) \sum_n \delta(t - nT) = \sum_n x(nT)\, \delta(t - nT)$
Rates	$\omega_s = 2\pi/T$ in rad/s, $f_s = 1/T$ in Hz, $\omega_s = 2\pi f_s$
Sampled spectrum	$X_p(j\omega) = \frac{1}{T} \sum_k X\big(j(\omega - k\omega_s)\big)$, at every rate
Guard band	$\omega_s - 2\omega_M$: positive, zero at the Nyquist rate, negative when the copies overlap
Sampling theorem	$X(j\omega) = 0$ for $\lvert\omega\rvert > \omega_M$ and $\omega_s > 2\omega_M$: the samples $x(nT)$ determine $x(t)$
Nyquist rate	$2\omega_M$ rad/s; the rate must be strictly above it
Band-pass sampling	$2\omega_H/n \le \omega_s \le 2\omega_L/(n-1)$, $n \le \omega_H/B$

TABLE 7.2 Reconstruction and aliasing.

RESULT	STATEMENT
Ideal filter	$H_r(j\omega) = T$ for $\lvert\omega\rvert < \omega_c$, 0 for $\lvert\omega\rvert > \omega_c$, with $\omega_M < \omega_c < \omega_s - \omega_M$
Interpolation	$x_r(t) = \sum_n x(nT)\,\frac{T\sin(\omega_c(t-nT))}{\pi(t-nT)}$; with $\omega_c = \pi/T$ the kernel is $\mathrm{sinc}(\pi t/T)$
Zero-order hold	$H_0(j\omega) = e^{-j\omega T/2}\,\frac{2\sin(\omega T/2)}{\omega}$, with $H_0(j0) = T$
First-order hold	$H_1(j\omega) = \frac{1}{T}\left[\frac{\sin(\omega T/2)}{\omega/2}\right]^2$, with $H_1(j0) = T$
Alias of a cosine	$\frac{\omega_s}{2} < \omega_0 < \omega_s$, $\omega_c = \frac{\omega_s}{2}$: $x_r(t) = \cos\big((\omega_s - \omega_0)t\big)$; in general $f_a = \lvert f_0 - kf_s\rvert \le f_s/2$
Anti-aliasing	A low-pass with cutoff $\omega_s/2$, placed before the sampler; with a transition band Δ, $f_s \ge 2(f_M + \Delta)$

TABLE 7.3 Discrete-time processing and rate change.

RESULT	STATEMENT
Frequency map	$\Omega = \omega T$; $\omega_s \mapsto 2\pi$, $\omega_s/2 \mapsto \pi$
Samples	$X_d(e^{j\Omega}) = \frac{1}{T}\sum_k X_c\big(j(\Omega - 2\pi k)/T\big)$
Equivalent system	$H_{\mathrm{eff}}(j\omega) = H_d(e^{j\omega T})$ for $\lvert\omega\rvert < \omega_s/2$, 0 beyond, for band-limited inputs
Quantization	$\Delta = 2/2^B$, $\lvert e\rvert \le \Delta/2$, noise power $\Delta^2/12$, $\mathrm{SNR} \approx 6.02B + 1.76$ dB
Sampled sequence	$X_p(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1} X\left(e^{j(\omega - 2\pi k/N)}\right)$; no aliasing when $\omega_M < \pi/N$
Decimation	$x_b[n] = x[nN]$, $X_b(e^{j\omega}) = X_p(e^{j\omega/N})$; low-pass to π/N first
Interpolation	$x_{(N)}[n] \leftrightarrow X_b(e^{jN\omega})$; low-pass of gain N, cutoff π/N
Rate change L/M	Up by L, one low-pass of gain L and cutoff $\min(\pi/L, \pi/M)$, down by M

> ⚠ ROWS WITH A CONDITION
>
> The sampling theorem needs a band-limited $x(t)$, and a signal of finite duration never is. The equivalent system holds only for band-limited inputs. Every sinc in these tables is the unnormalised $\mathrm{sinc}(\theta) = \sin\theta/\theta$.

What to carry forward

- An ideal sampler produces $x_p(t) = \sum_n x(nT)\,\delta(t - nT)$: an impulse at every nT whose weight is the sample $x(nT)$.
- One rate has two numbers: $\omega_s = 2\pi/T$ in rad/s and $f_s = 1/T$ in Hz. Check every period with $\omega_s T = 2\pi$.
- The sampled spectrum has a copy at every multiple of ω_s, scaled by $1/T$, at every rate.
- The guard band $\omega_s - 2\omega_M$ between the baseband and the first copy is positive, zero at the Nyquist rate, or negative.

- Aliasing is the overlap of neighbouring copies, which happens only when $\omega_s < 2\omega_M$. No filter after the sampler can undo it; a low-pass before it prevents it.
- The sampling theorem needs a band limit and $\omega_s > 2\omega_M$, strictly. A signal of finite duration is never band-limited.
- Ideal reconstruction is a low-pass of gain T with $\omega_M < \omega_c < \omega_s - \omega_M$; in time it adds one kernel per sample.
- The zero-order hold equals T at $\omega = 0$, sags in the band and leaks outside it. The first-order hold falls off like $1/\omega^2$, and its output is delayed by T.
- A cosine above $\omega_s/2$ comes back at the alias frequency. A wheel on film and stripes on a pixel grid follow the same rule.
- For band-limited inputs, sample, process and rebuild acts as $H_{\text{eff}}(j\omega) = H_d(e^{j\omega T})$ on $|\omega| < \omega_s/2$. Quantization adds a loss that only more bits reduce, about 6 dB per bit.
- Decimation stretches the spectrum by N and needs a low-pass first; interpolation compresses it by N and needs a low-pass after.

> ⊗ THE FOUR TRAPS, IN ONE PLACE
>
> Saying the copies disappear below the Nyquist rate: they never do, they overlap. Treating $\omega_s = 2\omega_M$ as safe: the admissible cutoff interval is empty there. Reading $2\pi/T$ in hertz: that is rad/s, and the error is a factor of 2π. Calling a hold output the reconstructed signal: a hold is an approximation with a measurable error.

When a signal is band-limited and the sampling rate satisfies the strict condition, its samples determine it uniquely. This is the bridge from continuous to discrete time: the continuous-time operations of the earlier chapters can be carried out on the samples, and the result converted back.

Exercises

7.1 A signal is sampled at $\omega_s = 8000\pi$ rad/s. Give T and f_s, and verify the period with a unit check.
Answer: $T = 2\pi/\omega_s = 1/4000$ s $= 0.25$ ms and $f_s = 1/T = 4000$ Hz. Check: $\omega_s T = 8000\pi \times 2.5 \times 10^{-4} = 2\pi$. A period of 0.25 s fails the same check by a factor of 1000.

7.2 $X(j\omega)$ peaks at 1 and $T = 0.2$ s. How tall is each copy in $X_p(j\omega)$, and where is the copy $k = 2$ centred when $T = 0.125$ s?
Answer: Each copy is scaled by $1/T = 1/0.2 = 5$. With $T = 0.125$ s, $\omega_s = 2\pi/0.125 = 16\pi$ rad/s, so the copy $k = 2$ is centred at $2\omega_s = 32\pi$ rad/s.

7.3 State in one sentence each what sampling does at every rate, and what happens only below the Nyquist rate.
Answer: At every rate sampling puts a copy of $X(j\omega)$ at every multiple of ω_s, scaled by $1/T$. Below the Nyquist rate those copies overlap and add, and that overlap is aliasing.

7.4 Show that the sampling theorem cannot be satisfied at $\omega_s = 2\omega_M$, and give a signal that demonstrates it.
Answer: The cutoff condition $\omega_M < \omega_c < \omega_s - \omega_M$ becomes $\omega_M < \omega_c < \omega_M$, which is empty. For $x(t) = \sin(\omega_M t)$ sampled at $\omega_s = 2\omega_M$ every sample is $\sin(\pi n) = 0$. For $\sin(50\pi t)$ at 100π rad/s, for example, every sample is 0.

7.5 What is the Nyquist rate of $\cos(300\pi t) + \cos(700\pi t)$, and what is ω_M for $x(t) = \left(\sin(50t)/(\pi t)\right)^2$?

Answer: The highest line is 700π rad/s, so the Nyquist rate is 1400π rad/s. Squaring convolves the rectangle on $|\omega| \le 50$ with itself, so the band doubles: $\omega_M = 100$ rad/s.

7.6 A triangular spectrum has peak $X_{\max} = 4000$ and is sampled at $\omega_s = 32000\pi$ rad/s. Find the height of one copy, and say why the area $A = 8000\pi$ is not the number to use.

Answer: $T = 2\pi/32000\pi = 1/16000$ s, so the copy height is $X_{\max}/T = 4000 \times 16000 = 6.4 \times 10^7$. The area of the convolution and the peak of the spectrum differ by 2π, since squaring in time convolves in frequency and divides by 2π; using A inflates every height by that factor.

7.7 $\omega_M = 3\pi$ and $\omega_s = 10\pi$ rad/s. Which of 2π, 5π and 8π is a working cutoff ω_c?

Answer: The cutoff needs $3\pi < \omega_c < 10\pi - 3\pi = 7\pi$, so only 5π works.

7.8 A zero-order hold has $T = 1$ ms. Where does $|H_0(j\omega)|$ first reach zero, and what is $|H_0|$ there relative to T at the band edge $\omega_s/2$?

Answer: The first zero is at $\omega = \omega_s = 2\pi/T = 2000\pi$ rad/s. At $\omega_s/2$, $|H_0| = 2T/\pi \approx 0.64\,T$.

7.9 $\cos(10\pi t)$ is sampled at $\omega_s = 16\pi$ rad/s and reconstructed with $\omega_c = 8\pi$. What comes out?

Answer: $10\pi > 8\pi$, so the baseband line is removed. The copy line $16\pi - 10\pi = 6\pi < 8\pi$ is kept, so $x_r(t) = \cos(6\pi t)$.

7.10 $x(t) = \cos(\pi t) + \cos(3\pi t)$ is sampled with $T = 2/5$ s and reconstructed with $\omega_c = \omega_s/2$. Find $x_r(t)$ and name the component that moved.

Answer: $\omega_s = 5\pi < 6\pi$, so aliasing occurs. Surviving lines: π from the baseband and $5\pi - 3\pi = 2\pi$ from the $k = 1$ copy, giving $x_r(t) = \cos(\pi t) + \cos(2\pi t)$. The component at 3π moved to 2π; the one at π passed untouched.

7.11 The chirp $\cos(2\pi \cdot 1000\, t^2)$ is sampled at $f_s = 3$ kHz. What pitch is heard at $t = 2$ s?

Answer: At $t = 2$ s the tone is at $2000 \times 2 = 4000$ Hz, and $|4 - 3| = 1$ kHz lies inside 0 to 1.5 kHz. The pitch heard is 1 kHz.

7.12 A wheel turns at 23 rev/s and is filmed at 24 frames/s. Stripes of period 1.0 mm lie over stripes of period 1.25 mm. What does the film show, and how far apart are the dark bands of the overlay?

Answer: $23 - 24 = -1$: the wheel seems to turn backwards at 1 rev/s. The spatial frequencies are 1 and $1/1.25 = 0.8$ cycle/mm, so $P = 1/(1 - 0.8) = 5$ mm.

7.13 A telephone line keeps speech up to 3.4 kHz and samples at 8 kHz. How wide may the transition band of its anti-aliasing filter be?

Answer: $\Delta = f_s/2 - 3.4 = 4 - 3.4 = 0.6$ kHz.

7.14 The digital differentiator runs with $T = 0.25$ s. What is $|H_d(e^{j\Omega})|$ at $\Omega = \pi/2$, and what peak does the output reach for a 700 Hz tone when $f_s = 1$ kHz?

Answer: $|\Omega|/T = (\pi/2)/0.25 = 2\pi$. The 700 Hz tone folds to $1000 - 700 = 300$ Hz first, so the output is the derivative of the alias, with peak $2\pi \cdot 300\ \mathrm{s}^{-1}$.

7.15 A converter is changed from 4 bits to 12 bits over the range -1 to 1. Give the largest rounding error before, and the change in SNR.

Answer: With 4 bits, $\Delta = 2/2^4 = 1/8$ and $|e| \le 1/16$. Eight more bits add about $8 \times 6.02 \approx 48$ dB.

7.16 The running sequence of Section 7.6, $\omega_M = \pi/4$, is decimated by $N = 2$. Where is the new band edge, and does it alias? A 44.1 kHz recording is converted to 48 kHz: which ratio L/M does it use?
Answer: The band edge moves to $N\omega_M = \pi/2 < \pi$, so there is no aliasing. The rate rises by $48000/44100 = 160/147$: up by $L = 160$, down by $M = 147$.

APPENDIX

A

Summary of formulas

A.1 Energy and power

TABLE A.1 Energy and power in continuous and discrete time.

QUANTITY	CONTINUOUS TIME	DISCRETE TIME
Total energy	$E_\infty = \int_{-\infty}^{\infty} \lvert x(t)\rvert^2 \mathrm{d}t$	$E_\infty = \sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2$
Average power	$P_\infty = \lim\limits_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \lvert x(t)\rvert^2 \mathrm{d}t$	$P_\infty = \lim\limits_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \lvert x[n]\rvert^2$
Energy signal	$E_\infty < \infty$, $P_\infty = 0$	same
Power signal	$E_\infty \to \infty$, $0 < P_\infty < \infty$	same

A.2 Operations and periodicity

TABLE A.2 Operations on the time axis, and the test for periodicity.

ITEM	STATEMENT
Shift	$x(t - t_0)$: delay if $t_0 > 0$, advance if $t_0 < 0$
Reversal	$x(-t)$, $x[-n]$
Scaling	$y(t) = x(at)$: compressed if $a > 1$, stretched if $0 < a < 1$; support divided by a
Combination	$x(at - b)$: shift by b first, then scale by a
Periodicity	$x(t) = x(t + T)$; $x[n] = x[n + N]$ with N an integer
Fundamental frequency	$\omega_0 = 2\pi/T_0 = 2\pi/N_0$
Even and odd	$\mathcal{E}\mathrm{v}\{x\} = \frac{1}{2}[x(t) + x(-t)]$, $\mathcal{O}\mathrm{dd}\{x\} = \frac{1}{2}[x(t) - x(-t)]$

A.3 Impulses and steps

TABLE A.3 The unit impulse and the unit step in continuous and discrete time.

ITEM	CONTINUOUS TIME	DISCRETE TIME
Step and impulse	$\delta(t) = \frac{\mathrm{d}}{\mathrm{d}t}u(t)$, $u(t) = \int_{-\infty}^{t} \delta(\tau)\mathrm{d}\tau$	$\delta[n] = u[n] - u[n-1]$, $u[n] = \sum_{k=0}^{\infty} \delta[n-k]$
Sampling	$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$	$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$
Sifting	$x(t_0) = \int_{-\infty}^{\infty} x(t)\delta(t-t_0)\mathrm{d}t$	$x[n_0] = \sum_n x[n]\delta[n-n_0]$
Representation	$x(t) = \int x(\tau)\delta(t-\tau)\mathrm{d}\tau$	$x[n] = \sum_k x[k]\delta[n-k]$

A.4 Complex exponentials

TABLE A.4 Complex exponentials in continuous and discrete time, and their periods.

ITEM	STATEMENT
Continuous time	$x(t) = Ce^{at}$; with $C = Ae^{j\theta}$ and $a = r + j\omega_0$: $x(t) = Ae^{rt}e^{j(\omega_0 t+\theta)}$
Period	$T_0 = 2\pi/\omega_0$, always periodic for $\omega_0 \neq 0$
Discrete time	$x[n] = C\alpha^n$ with $\alpha = e^{\beta}$; growth boundary at $\lvert\alpha\rvert = 1$
Period	$N = \frac{2\pi}{\omega_0}k$; periodic only if $\omega_0/2\pi \in \mathbb{Q}$
Frequency wrap-around	$e^{j(\omega_0+2\pi)n} = e^{j\omega_0 n}$

A.5 Systems and convolution

TABLE A.5 System properties, in general and for an LTI system in terms of h.

PROPERTY	GENERAL CRITERION	LTI CRITERION IN TERMS OF h
Memoryless	output at t uses only input at t	$h(t) = a\delta(t)$ or $h[n] = a\delta[n]$
Invertible	distinct inputs give distinct outputs	$h * g = \delta$ for some g
Causal	output uses only $\tau \leq t$	$h(t) = 0$ for $t < 0$; $h[n] = 0$ for $n < 0$
BIBO stable	bounded input gives bounded output	$\int \lvert h\rvert\, \mathrm{d}t < \infty$; $\sum_k \lvert h[k]\rvert < \infty$
Time invariant	$x(t-t_0) \to y(t-t_0)$	—
Linear	$ax_1 + bx_2 \to ay_1 + by_2$	—

CONVOLUTION

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,\mathrm{d}\tau$$

Convolution is commutative, distributive, and associative. The output support is the sum of the input supports. The total areas, or the total sums in discrete time, multiply.

A.7 Fourier series properties

A periodic signal of period T_0 with $\omega_0 = 2\pi/T_0$, or a periodic sequence of period N with $\omega_0 = 2\pi/N$. In both columns $x \leftrightarrow a_k$ and $y \leftrightarrow b_k$.

TABLE A.6 Fourier series properties in continuous and discrete time.

PROPERTY	CONTINUOUS TIME	DISCRETE TIME
Linearity	$Ax(t) + By(t) \leftrightarrow Aa_k + Bb_k$	$Ax[n] + By[n] \leftrightarrow Aa_k + Bb_k$
Time shift	$x(t - t_0) \leftrightarrow a_k e^{-jk\omega_0 t_0}$	$x[n - n_0] \leftrightarrow a_k e^{-jk(2\pi/N)n_0}$
Frequency shift	$e^{jM\omega_0 t}x(t) \leftrightarrow a_{k-M}$	$e^{jM(2\pi/N)n}x[n] \leftrightarrow a_{k-M}$
Conjugation	$x^*(t) \leftrightarrow a^*_{-k}$	$x^*[n] \leftrightarrow a^*_{-k}$
Time reversal	$x(-t) \leftrightarrow a_{-k}$	$x[-n] \leftrightarrow a_{-k}$
Time scaling	$x(\alpha t)$, $\alpha > 0$: a_k, period T_0/α	$x_{(m)}[n]$: $\frac{1}{m}a_k$, period mN
Periodic convolution	$\int_{T_0} x(\tau)y(t-\tau)\mathrm{d}\tau \leftrightarrow T_0 a_k b_k$	$\sum_{r=\langle N\rangle} x[r]y[n-r] \leftrightarrow N a_k b_k$
Multiplication	$x(t)y(t) \leftrightarrow \sum_{\ell=-\infty}^{\infty} a_\ell b_{k-\ell}$	$x[n]y[n] \leftrightarrow \sum_{\ell=\langle N\rangle} a_\ell b_{k-\ell}$
Derivative or difference	$\mathrm{d}x/\mathrm{d}t \leftrightarrow jk\omega_0 a_k$	$x[n] - x[n-1] \leftrightarrow (1 - e^{-jk(2\pi/N)})a_k$
Integral or running sum	$\int_{-\infty}^{t} x\,\mathrm{d}\tau \leftrightarrow \frac{a_k}{jk\omega_0}$, only if $a_0 = 0$	$\sum_{r=-\infty}^{n} x[r] \leftrightarrow \frac{a_k}{1-e^{-jk(2\pi/N)}}$, only if $a_0 = 0$
Real signal	$a_k = a^*_{-k}$; $\lvert a_k\rvert$ even, $\angle a_k$ odd	same
Real and even	a_k real and even	same
Real and odd	a_k purely imaginary and odd	same
Even-odd parts	$\mathcal{E}\mathrm{v}\{x\} \leftrightarrow \mathrm{Re}\{a_k\}$, $\mathcal{O}\mathrm{dd}\{x\} \leftrightarrow j\,\mathrm{Im}\{a_k\}$	same
Parseval	$\frac{1}{T_0}\int_{T_0} \lvert x\rvert^2 \mathrm{d}t = \sum_k \lvert a_k\rvert^2$	$\frac{1}{N}\sum_{n=\langle N\rangle} \lvert x\rvert^2 = \sum_{k=\langle N\rangle} \lvert a_k\rvert^2$

The two columns differ in three rows because discrete-time coefficients repeat, $a_k = a_{k+N}$. For multiplication, the discrete-time sum therefore covers one period instead of all integers. Time expansion carries the factor $\frac{1}{m}$ because the average uses m times as many samples. Discrete time also uses a first difference where continuous time uses a derivative. At $k = 0$, integration or accumulation requires zero mean; otherwise the result grows without bound.

A.8 Continuous-time Fourier transform

TABLE A.7 Properties of the continuous-time Fourier transform.

PROPERTY	STATEMENT
Linearity	$ax_1(t) + bx_2(t) \leftrightarrow aX_1(j\omega) + bX_2(j\omega)$
Time shift	$x(t - t_0) \leftrightarrow e^{-j\omega t_0} X(j\omega)$
Frequency shift	$e^{j\omega_0 t} x(t) \leftrightarrow X(j(\omega - \omega_0))$
Conjugation	$x^*(t) \leftrightarrow X^*(-j\omega)$
Time reversal	$x(-t) \leftrightarrow X(-j\omega)$
Time and frequency scaling	$x(at) \leftrightarrow \frac{1}{\lvert a\rvert} X(j\omega/a)$
Convolution	$x(t) * h(t) \leftrightarrow X(j\omega)H(j\omega)$
Multiplication	$x(t)y(t) \leftrightarrow \frac{1}{2\pi} X(j\omega) * Y(j\omega)$
Differentiation in time	$\mathrm{d}^n x/\mathrm{d}t^n \leftrightarrow (j\omega)^n X(j\omega)$
Integration	$\int_{-\infty}^{t} x(\tau)\mathrm{d}\tau \leftrightarrow \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$
Differentiation in frequency	$t\,x(t) \leftrightarrow j\,\mathrm{d}X(j\omega)/\mathrm{d}\omega$
Duality	$X(t) \leftrightarrow 2\pi x(-\omega)$
Real signal	$X(-j\omega) = X^*(j\omega)$; $\lvert X\rvert$ even, $\angle X$ odd
Real and even	$X(j\omega)$ real and even
Real and odd	$X(j\omega)$ purely imaginary and odd
Even-odd parts	$\mathcal{E}\mathrm{v}\{x\} \leftrightarrow \mathrm{Re}\{X\}$, $\mathcal{O}\mathrm{dd}\{x\} \leftrightarrow j\,\mathrm{Im}\{X\}$
Parseval	$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2 \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lvert X(j\omega)\rvert^2 \mathrm{d}\omega$

TABLE A.8 Continuous-time Fourier transform pairs.

SIGNAL	TRANSFORM
$\delta(t)$	1
$\delta(t-t_0)$	$e^{-j\omega t_0}$
$u(t)$	$\frac{1}{j\omega}+\pi\delta(\omega)$
$e^{-at}u(t)$, $a>0$	$\frac{1}{a+j\omega}$
$te^{-at}u(t)$, $a>0$	$\frac{1}{(a+j\omega)^2}$
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$, $a>0$	$\frac{1}{(a+j\omega)^n}$
$e^{-a\lvert t\rvert}$, $a>0$	$\frac{2a}{a^2+\omega^2}$
1 on $\lvert t\rvert<T_1$, 0 beyond	$\frac{2\sin(\omega T_1)}{\omega}=2T_1\,\mathrm{sinc}(\omega T_1)$
$\frac{\sin(Wt)}{\pi t}=\frac{W}{\pi}\,\mathrm{sinc}(Wt)$	1 on $\lvert\omega\rvert<W$, 0 beyond
1	$2\pi\delta(\omega)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\cos\omega_0 t$	$\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$
$\sin\omega_0 t$	$\frac{\pi}{j}\delta(\omega-\omega_0)-\frac{\pi}{j}\delta(\omega+\omega_0)$
$\sum_k a_k e^{jk\omega_0 t}$	$\sum_k 2\pi a_k\delta(\omega-k\omega_0)$
Square wave: 1 on $\lvert t\rvert<T_1$, 0 on $T_1<\lvert t\rvert\le T/2$	$\sum_k \frac{2\sin(k\omega_0 T_1)}{k}\delta(\omega-k\omega_0)$
$\sum_k \delta(t-kT)$	$\frac{2\pi}{T}\sum_k \delta(\omega-\frac{2\pi k}{T})$

This table uses the unnormalised definition $\mathrm{sinc}(\theta)=\sin\theta/\theta$. The normalised convention divides the argument by π. Convert the argument when moving between conventions; otherwise the result loses a factor of π. The exponential pairs require $a>0$.

A.9 Discrete-time Fourier transform

TABLE A.9 Properties of the discrete-time Fourier transform.

PROPERTY	STATEMENT
Linearity	$ax_1[n] + bx_2[n] \leftrightarrow aX_1(e^{j\omega}) + bX_2(e^{j\omega})$
Time shift	$x[n-n_0] \leftrightarrow e^{-j\omega n_0}X(e^{j\omega})$
Frequency shift	$e^{j\omega_0 n}x[n] \leftrightarrow X(e^{j(\omega-\omega_0)})$
Conjugation	$x^*[n] \leftrightarrow X^*(e^{-j\omega})$
Time reversal	$x[-n] \leftrightarrow X(e^{-j\omega})$
Time expansion	$x_{(k)}[n] \leftrightarrow X(e^{jk\omega})$
Convolution	$x[n] * h[n] \leftrightarrow X(e^{j\omega})H(e^{j\omega})$
Multiplication	$x[n]y[n] \leftrightarrow \frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})\mathrm{d}\theta$
Differencing in time	$x[n] - x[n-1] \leftrightarrow (1-e^{-j\omega})X(e^{j\omega})$
Accumulation	$\sum_{m=-\infty}^{n} x[m] \leftrightarrow \frac{X(e^{j\omega})}{1-e^{-j\omega}} + \pi X(e^{j0})\sum_k \delta(\omega - 2\pi k)$
Differentiation in frequency	$n\,x[n] \leftrightarrow j\,\mathrm{d}X(e^{j\omega})/\mathrm{d}\omega$
Real sequence	$X(e^{-j\omega}) = X^*(e^{j\omega})$; $\lvert X\rvert$ even, $\angle X$ odd
Real and even	$X(e^{j\omega})$ real and even
Real and odd	$X(e^{j\omega})$ purely imaginary and odd
Even-odd parts	$\mathcal{E}\mathrm{v}\{x\} \leftrightarrow \mathrm{Re}\{X\}$, $\mathcal{O}\mathrm{dd}\{x\} \leftrightarrow j\,\mathrm{Im}\{X\}$
Parseval	$\sum_n \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{2\pi} \lvert X(e^{j\omega})\rvert^2\mathrm{d}\omega$

TABLE A.10 Discrete-time Fourier transform pairs.

SEQUENCE	TRANSFORM
$\delta[n]$	1
$\delta[n-n_0]$	$e^{-j\omega n_0}$
$u[n]$	$\frac{1}{1-e^{-j\omega}} + \pi \sum_k \delta(\omega - 2\pi k)$
$a^n u[n]$, $\lvert a\rvert < 1$	$\frac{1}{1-ae^{-j\omega}}$
$(n+1)a^n u[n]$, $\lvert a\rvert < 1$	$\frac{1}{(1-ae^{-j\omega})^2}$
$\frac{(n+r-1)!}{n!(r-1)!} a^n u[n]$, $\lvert a\rvert < 1$	$\frac{1}{(1-ae^{-j\omega})^r}$
$a^{\lvert n\rvert}$, $\lvert a\rvert < 1$	$\frac{1-a^2}{1-2a\cos\omega+a^2}$
1 on $\lvert n\rvert \le N_1$, 0 beyond	$\frac{\sin(\omega(N_1+\frac{1}{2}))}{\sin(\omega/2)}$
$\frac{\sin(Wn)}{\pi n}$, $0 < W < \pi$	1 on $\lvert\omega\rvert \le W$, 0 on $W < \lvert\omega\rvert \le \pi$
1	$2\pi \sum_k \delta(\omega - 2\pi k)$
$e^{j\omega_0 n}$	$2\pi \sum_k \delta(\omega - \omega_0 - 2\pi k)$
$\cos \omega_0 n$	$\pi \sum_k [\delta(\omega - \omega_0 - 2\pi k) + \delta(\omega + \omega_0 - 2\pi k)]$
$\sin \omega_0 n$	$\frac{\pi}{j} \sum_k [\delta(\omega - \omega_0 - 2\pi k) - \delta(\omega + \omega_0 - 2\pi k)]$
$\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$	$2\pi \sum_k a_k \delta(\omega - \frac{2\pi k}{N})$
$\sum_k \delta[n-kN]$	$\frac{2\pi}{N} \sum_k \delta(\omega - \frac{2\pi k}{N})$

Every transform in this section is periodic in ω with period 2π because the time index is discrete. A periodic sequence has discrete spectral coefficients, while an aperiodic sequence has a continuous spectrum. In each case, discreteness in one domain produces periodicity in the other.

A.10 Symbols

TABLE A.11 Symbols and their meanings.

SYMBOL	MEANING
$x(t)$, $x[n]$	input signal, continuous and discrete time
$y(t)$, $y[n]$	output signal
$h(t)$, $h[n]$	impulse response of a linear time-invariant system
$\delta(t)$, $\delta[n]$	unit impulse
$u(t)$, $u[n]$	unit step
E_∞, P_∞	total energy and average power over all time
T_0, N_0	fundamental period
ω_0	fundamental angular frequency, rad/s or rad/sample
$*$	convolution
j	imaginary unit, $j^2 = -1$

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VERSION	DATE	CHANGES
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